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A CONSTRUCTIVE NEGATION DEFINED WITH A NEGATION CONNECTIVE FOR LOGICS INCLUDING Bp_+

Abstract

The concept of constructive negation we refer to in this paper is (minimally) intuitionistic in character (see [1]). The idea is to understand the negation of a proposition A as equivalent to A implying a falsity constant of some sort. Then, negation is introduced either by means of this falsity constant or, as in this paper, by means of a propositional connective defined with the constant. But, unlike intuitionisitc logic, the type of negation we develop here is, of course, devoid of paradoxes of relevance.

1. Introduction

We explain what we understand by "constructive negation". Next, we comment some previous results and state the aim of the paper.

1.1. The concept of constructive negation

The concept of constructive negation we refer to in this paper is (minimally) intuitionistic in character (see [1]). The idea is to understand the negation of a proposition A as equivalent to A implying a falsity constant of some sort. Then, negation is introduced either by means of this falsity constant or, as in this paper, by means of a propositional connective defined with the constant. But, unlike intuitionisitc logic, the type of negation we develop here is, of course, devoid of paradoxes of relevance.

This concept of constructive negation can essentially be understood in two different senses:

a. The first one coincides with the negation characterisitic of Johansson's minimal intuitionistic logic that can intuitively be described by the presence of the weak versions of "double negation" ($A \to \neg\neg A$), "contraposition" ($(A \to \neg B) \to (B \to \neg A)$) and "reductio" ($(A \to B) \to [(A \to \neg B) \to \neg A]$).
b. In the second one, the idea is to add a falsity constant F to a given positive logic S_+. Next, one defines $\neg A = A \to F$ but no new axioms are added to S_+. Then, it is the positive logic S_+ that provides its, so to speak, underlying "concept of negation". If S_+ is positive intuitionistic logic I_+, senses a and b are of course equivalent. But, what happens if S_+ is a weaker postive logic?

In what follows sense a will be labelled "Johansson negation" and sense b "minimal negation".

1.2. Previous results

Let a "quasi-Johansson negation" be defined by the presence of the (weak) double negation and contraposition axioms together with "specialized reductio" ($(A \to \neg A) \to \neg A$) instead of the "full redutio axioms" ($(A \to B) \to [(A \to \neg B) \to \neg A]$, $(A \to \neg B) \to [(A \to B) \to \neg A]$). In [3], it is shown how to introduce a minimal and a quasi-Johansson negation in the basic positive logic B_+ of Routley and Meyer (see [4]) by using a falsity constant. As B_+ is the weakest logic definable in the ternary relational semantics, it is actually shown how to introduce both types of negation in any logic representable with the relational ternary semantics. In that same paper it is argued that the full reductio axioms cannot be introduced in B_+, the resources of this logic being insufficient to prove the corresponding semantical conditions for the axioms. Moreover, as it is dicussed in [2], this seems to be so even in the case of the strong full non-constructive axioms. Now, in [3] it is proved that if the prefixing axiom ($(B \to C) \to [(A \to B) \to (A \to C)]$) is added to B_+, the full reductio axioms can be introduced in the resulting logic called Bp_+.

1.3. Aim of the paper

The aim of this paper is to develop the logic suggested in [3] and commented above. We will show how to introduce a Johansson negation in any logic including Bp_+ by using a negation connective. A distinctive feature of the resulting logic will be the presence of the "permuted contraposition axioms" ($\neg B \to [(A \to B) \to \neg A]$ and $B \to [(A \to \neg B) \to \neg A]$) that are not theorems of B_{mr} (B_+ with the quasi-Johansson negation) defined in [3]. (The reason that explains the absence of the full reductio axioms in B_{mr} is also an explanation of the absence of the permuted contraposition axioms). In particular, the structure of the paper is the following. In §2, the logic Bp_+ is defined. In §3, §4, we introduce the logic Bpc which is the result of introducing the contraposition and permuted contraposition axioms together with double negation. In §5, the logic Bpcr is defined. It is the result of adding the full reductio axioms to Bpc and, as it was remarked above, it is considerable stronger than the extension of Bp_+ suggested in [3]. Finally, in §6, we briefly comment on the relationship between Bpcr and modal and relevance logics.

2. The logic Bp_+

The logic Bp_+ is the result of adding the prefixing axiom (A2) to B_+. That is, Bp_+ is axiomatized with

A1. $A \to A$
A2. $(B \to C) \to [(A \to B) \to (A \to C)]$
A3. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$
A4. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of inference are *Modus ponens* (MP) (if $\vdash A \to B$ and $\vdash A$, then $\vdash B$), *Adjunction* (Adj) (if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$) and *Suffixing* (Suf) (if $\vdash A \to B$, then $\vdash (B \to C) \to (A \to C)$).

Note. The logic B_+ is the result of dropping the prefixing axiom A2 and adding the prefixing rule (Pref): (if $\vdash B \to C$, then $\vdash (A \to B) \to (A \to C)$).

A Bp_+ *model* is a quadruple $< K, O, R, \models>$ where K is a set, O a subset of K and R a ternary relation defined on K subject to the following definitions and postulates for all $a, b, c, d \in K$:

d1. $a \leq b =_{df} (\exists x \in O)\ Rxab$
d2. $R^2abcd =_{df} (\exists x \in K)\ [Rabx\ \&\ Rxcd]$
P1. $a \leq a$
P2. $(a \leq b\ \&\ Rbcd) \Rightarrow Racd$
P3. $R^2abcd \Rightarrow (\exists x \in K)\,[Rbcx\ \&\ Raxd]$

$\models$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositonal variables p, wffs A, B and $a, b, c \in K$:

(i). $(a \models p\ \&\ a \leq b) \Rightarrow b \models p$
(ii). $a \models A \vee B$ iff $a \models A$ or $a \models B$
(iii). $a \models A \wedge B$ iff $a \models A$ and $a \models B$
(iv). $a \models A \rightarrow B$ iff for all $b, c \in K$ $(Rabc\ \&\ b \models A) \Rightarrow c \models B$

A formula A is *valid* ($\models_{Bp_+} A$) iff $a \models A$ for all $a \in O$ in all models.

NOTE. B_+ models are exactly as Bp_+ models but without the postulate P3.

In e.g, [3], it is proved that A is a theorem of Bp_+ iff A is Bp_+ valid.

3. The logic Bpc

We add to the positive propositional language the unary connective $\neg$. Then, the logic Bpc (Bp_+ with (weak) contraposition and (weak) double negation) is axiomatized by adding to Bp_+ the axioms:

A7. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
A8. $B \rightarrow [(A \rightarrow \neg B) \rightarrow \neg A]$
A9. $(A \rightarrow B) \rightarrow [(B \rightarrow \neg C) \rightarrow (A \rightarrow \neg C)]$

A7 is a form of weak contraposition (other forms are A8 and T2, T3 below), A8 is permuted A7 and A9 is a restricted version of the suffixing axiom

$$(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$$

to the case in which C is a negative formula. The following, for example, are theorems of Bpc (a sketch of the proof is provided at the right side of the theorem)

T1.	$A \to \neg\neg A$	A7
T2.	$(A \to B) \to (\neg B \to \neg A)$	A7, T1
T3.	$\neg B \to [(A \to B) \to \neg A]$	A8, T1
T4.	$A \to [(A \to \neg B) \to \neg B]$	A7, A8
T5.	$[A \to (B \to \neg C)] \to [B \to (A \to \neg C)]$	A9, T4
T6.	$B \to [[A \to (B \to \neg C)] \to (A \to \neg C)]$	T4

Theorems T4-T6 are restricted versions of *assertion*

$$A \to [(A \to C) \to C]$$

permutation

$$[A \to (B \to C)] \to [B \to (A \to C)]$$

and *conditioned modus ponens*

$$B \to [[A \to (B \to C)] \to (A \to C)]$$

to the case in which C is a negative formula (in §6, it is proved that the unrestricted version are not provable). We note that A9, T4, T5 and T6 can be generalized:

A9g. $(A \to B) \to \{[B \to [C_1 \to ((...(C_n \to \neg D)\,...))]] \to$
$[A \to [C_1 \to ((...(C_n \to \neg D)...))]]\}$
T4g. $A \to \{[A \to [B_1 \to ((...(B_n \to \neg C)...))]] \to$
$[B_1 \to ((...(B_n \to \neg C)...))]\}$
T5g. $\{A \to [B_1 \to ((...(B_n \to \neg C)...))]\} \to$
$\{B_1 \to [A \to [B_2 \to ((...(B_n \to \neg C)...))]]\}$
T6g. $B_1 \to \{[A \to [B_1 \to ((...(B_n \to \neg C)...))]] \to$
$[A \to [B_2 \to ((...(B_n \to \neg C)...))]]\}$

These generalized versions are proved as follows. Suppose we have A9g $(n = k - 1)$,T4g$(n = k - 1)$ and T5g$(n = k - 1)$. Then prove

i. T5g$(n = k)$ with A9g $(n = k - 1)$,T4g$(n = k - 1)$.
ii. T4g$(n = k)$ with T4g $(n = k - 1)$,T5g$(n = k)$.

iii. A9g($n = k$) with A9g ($n = k - 1$),T5g($n = k$).
iv. T6g($n = k$) with T4g($n = k - 1$).

For example, let us prove iii:

PROOF.

1. $(A \to B) \to \{[B \to [C_2 \to ((...(C_k \to \neg D)...))]] \to [A \to [C_2 \to ((...(C_k \to \neg D)...))]]\}$ A5g($n = k - 1$)

By A2

2. $(A \to B) \to \{[C_1 \to [B \to [C_2 \to ((...(C_k \to \neg D)...))]]] \to [C_1 \to [A \to [C_2 \to ((...(C_n \to \neg D)...))]]]\}$

Now, by applying T5g($n = k$)

3. $(A \to B) \to \{[B \to [C_1 \to [C_2 \to ((...(C_k \to \neg D)...))]]] \to [A \to [C_1 \to [C_2 \to ((...(C_k \to \neg D)...))]]]\}$

□

Other characteristic theorems of Bpc are:

T7.	$\neg A \to [A \to \neg(A \to A)]$	A7, T13
T8.	$[A \to \neg(A \to A)] \to \neg A$	A1, A8
T9.	$(\neg A \wedge \neg B) \leftrightarrow (\neg A \vee B)$	A7, T2
T10.	$(\neg A \vee \neg B) \to \neg(A \wedge B)$	T2

4. Semantics for Bpc

A *Bpc model* is a quintuple $< K, O, S, R, \models >$ where $< K, O, R, \models >$is a Bp_+ model and S is a subset of K such that $S \cap O \neq \emptyset$. The following clause and postulates are also added:

(v). $a \models \neg A$ iff for all $b, c \in K$, $(Rabc \ \& \ c \in S) \Rightarrow b \not\models A$
PA7. $R^2abcd \ \& \ d \in S \Rightarrow (\exists x \in S) \ R^2acbx$
PA8. $R^2abcd \ \& \ d \in S \Rightarrow (\exists x \in S) \ R^2bcax$
PA9. $R^3abcde \ \& \ e \in S \Rightarrow (\exists x, \ y \in K)(\exists z \in S) \ [Racx \ \& \ Rbxy \ \& \ Rydz]$

where

$$R^3abcde =_{df} (\exists x, y \in K) \ [Rabx \ \& \ Rxcy \ \& \ Ryde]$$

A is *Bpc valid* iff $a \models A$ for all $a \in O$ in all models.

Note. As it is known, the definition of minimal negation in the binary relational semantics (Kripke semantics) can be formulated as follows:

$a \models \neg A$ iff for all consistent points, if Rab, then $b \not\models A$

Now, clause (v) is the translation of this definition in the ternary relational semantics.

Semantic consistency (soundness) of Bpc follows, given that of Bp_+, just by proving that A7, A8 and A9 are valid: use PA7, PA8 and PA9, respectively.

As for completeness, the canonical model is the structure $< K^C$, O^C, S^C, R^C, $\models^C >$ where K^C, O^C, R^C and $\models^C$are defined as it is customary in relevance logics (see e.g, [4]) and S^C (that is, canonical S) is interpreted as the set of all consistent theories (a *theory* is a set of formulas closed under adjunction and provable entailment; a theory is *inconsistent* iff the negation of a theorem belongs to it). Then, given the completeness of Bp_+, we just have to prove that clause (v) and PA7, PA8 and PA9 hold canonically. Thus, we prove

Proposition 1. *PA7, PA8 and PA9 hold canonically, i.e,*

1. $R^{C2}abcd$ & $d \in S^C \Rightarrow \left(\exists x \in S^C\right)$ $R^{C2}acbx$
2. $R^{C2}abcd$ & $d \in S^C \Rightarrow \left(\exists x \in S^C\right)$ $R^{C2}bcax$
3. $R^{C3}abcde$ & $e \in S^C \Rightarrow \left(\exists x,\ y \in K^C\right) \left(\exists z \in S^C\right)$ $[R^C acx$ & $R^C bxy$ & $R^C ydz]$

It is clear that this proposition follows from the following lemma where K^T is the set of all theories and R^T is the extension of R^C to all theories:

Lemma 2.

1. *Let a, b,$c \in K^T$, d a consistent member in K^T and $R^{T2}abcd$. Then, there is some $x \in S^C$ such that $R^{T2}acbx$.*
2. *Let a, b,$c \in K^T$, d a consistent member in K^T and $R^{T2}abcd$. Then, there is some $x \in S^C$ such that $R^{T2}bcax$.*
3. *Let a, b,c, $d \in K^T$, e a consistent member in K^T and $R^{T3}abcde$. Then, there are x, $y \in K^T$ and $z \in S^C$ such that $R^T acx$, $R^T bxy$ and $R^T ydz$.*

Let us prove, for example, that iii holds (proof of i and ii are similar):

Proof. Let a, b,c, $d \in K^T$, e a consistent member in K^T and $R^{T3}abcde$, i.e, $R^T abx$, $R^T xcy$ and $R^T yde$ for some x, $y \in K^T$. We prove that there are z, $u \in K^T$ and $w \in S^C$ such that $R^T acz$, $R^T bzu$, $R^T udw$. Define the theories $z = \{B \mid A \rightarrow B \in a$ and $A \in c\}$, $u = \{B \mid A \rightarrow B \in b$ and $A \in z\}$ and $w\prime = \{B \mid A \rightarrow B \in u$ and $A \in d\}$ such that $R^T acz$, $R^T bzu$ and $R^T udw\prime$. We prove that $w\prime$ is consistent. Suppose it is not. Then, for some theorem A, $\neg A \in w\prime$. By definitions, $B \rightarrow \neg A \in u$, $B \in d$, $C \rightarrow (B \rightarrow \neg A) \in b$, $C \in z$, $D \rightarrow C \in a$, $D \in c$ for some wff B, C, D. Now, $\vdash (D \rightarrow C) \rightarrow [[C \rightarrow (B \rightarrow \neg A)] \rightarrow [D \rightarrow (B \rightarrow \neg A)]]$ is a theorem (A5g($n = 1$)). So, $[C \rightarrow (B \rightarrow \neg A)] \rightarrow [D \rightarrow (B \rightarrow \neg A)] \in a$ and by $R^T abx$, $D \rightarrow (B \rightarrow \neg A) \in x$; by $R^T xcy$, $B \rightarrow \neg A \in y$ and finally, $\neg A \in e$ by $R^T yde$ contradicting the consistency of e. Therefore, $w\prime$ is consistent. Next, $w\prime$ is extended to a prime theory w such that $R^T udw$. Now, parts i and ii of lemma 1 are proved similarly by using T5 and T6, respectively.□

Next, we prove

Proposition 3. *Clause (v) holds canonically.*

Proof.

1. If $\neg A \in a$, then $(R^C abc \;\&\; c \in S^C) \Rightarrow A \notin b$.
   Suppose $\neg A \in a$, $R^C abc$, $c \in S^C$ and, by reductio, $A \in b$, By T7, $A \rightarrow \neg(A \rightarrow A) \in a$. So, $\neg(A \rightarrow A) \in c$ by $R^C abc$ and $A \in b$. But c would be inconsistent.
2. If $\neg A \notin a$, then there are $b \in K^C$, $c \in S^C$ such that $R^C abc$ and $A \in b$.
   First, let us suppose that $\neg A \notin a$. Then define the theories $x = \{B \mid\vdash A \rightarrow B\}$ and $y = \{B \mid C \rightarrow B \in a$ and $C \in x\}$. It is easy to show that $R^T axy$ and $A \in x$ ($\vdash A \rightarrow A$). Next, we prove that y is consistent. If y is not consistent, then $\neg B \in y$, B being a theorem of Bpc. By definitions, $C \rightarrow \neg B \in a$, $C \in x$, $\vdash A \rightarrow C$ for some wff C. By Suf, $\vdash (C \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)$. Then, $A \rightarrow \neg B \in a$. Now, by A8, $\vdash (A \rightarrow \neg B) \rightarrow \neg A$ because B is a theorem. In consequence, $\neg A \in a$ which contradicts the hypothesis. Therefore, y is consistent. Finally, x and y are extended to prime theories b, c such that $R^C abc$ and $A \in b$.

□

5. The logic Bpcr

The logic Bpcr (Bpc plus (weak) reductio) is axiomatized by adding to Bpc:

A10. $(A \to B) \to [(A \to \neg B) \to \neg A]$

The following (in addition to T1-T10) are, for instance, theorems of Bpcr:

T11.	$(A \to \neg A) \to \neg A$	A10
T12.	$(A \to \neg B) \to [(A \to B) \to \neg A]$	A10, T5
T13.	$[A \to (A \to \neg B)] \to (A \to \neg B)$	A7, T5, T11
T14.	$[A \to (B \to \neg C)] \to [(A \to B) \to (A \to \neg C)]$	T5g($n = 2$), T13
T15.	$(A \to B) \to [[A \to (B \to \neg C)] \to (A \to \neg C)]$	A9, T13
T16.	$[A \to (B \to \neg C)] \to [(A \wedge B) \to \neg C]$	T15
T17.	$(A \wedge B) \to (A \to \neg B)$	A7, T4, T16
T18.	$(A \to \neg B) \to \neg (A \wedge B)$	A7, T17
T19.	$(A \to B) \to \neg (A \wedge \neg B)$	T1, T18
T20.	$(A \wedge \neg B) \to \neg (A \to B)$	A7, T19
T21.	$\neg (A \wedge \neg A)$	T19

We note that T13-T16 are restrictions of, respectively, *contraction*

$$[A \to (A \to C)] \to (A \to C)$$

self-distribution of the conditional

$$[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$$

permuted self-distribution of the conditional

$$(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$$

and *importation*

$$[A \to (B \to C)] \to [(A \wedge B) \to C]$$

to the case in which C is a negative formula (it is proved in §6 that the unrestricted versions are not provable).

Now, T13-T16 can be generalized:

T13g. $\{A \to [A \to [B_1 \to ...((...(B_n \to \neg C)...))]]\} \to$
$[A \to [B_1 \to ((...(B_n \to \neg C)...))]]$

T14g. $\{A \to [B_1 \to ((...(B_n \to \neg C)...))]\} \to$
$\{(A \to B_1) \to [A \to [B_2 \to ((...(B_n \to \neg C)...))]]\}$

T15g. $(A \to B_1) \to \{[A \to [B_1 \to ((...(B_n \to \neg C)...))]] \to$
$[A \to [B_2 \to ((...(B_n \to \neg C)...))]]\}$

T16g. $\{A \to [B_1 \to ((...(B_n \to \neg C)...))]\} \to$
$\{(A \wedge B_1) \to [B_2 \to ((...(B_n \to \neg C)...))]\}$

These generalized theorems are proved as follows:

Proof. T13g$(n = k)$: T5g$(n = k)$, T5g$(n = k + 1)$, T13g$(n = k - 1)$.
T14g$(n = k)$: T5g$(n = k + 1)$, T13g$(n = k - 1)$.
T15g$(n = k)$: T13g$(n = k - 1)$.
T16g$(n = k)$: T5g$(n = k)$.
These proofs are developed similarly as the proof of A9g$(n = k)$. □

6. Semantics for Bpcr

Models for Bpcr are defined similarly as those for Bpcr but with the addition of the postulate

PA9. $R^2abcd \ \& \ d \in S \Rightarrow (\exists x, y \in K)\,(\exists z \in S)\,[Racx \ \& \ Rbcy \ \& \ Ryxz]$

This postulate is used in showing the validity of A10 thus proving the semantic consistency (soundness) of Bpcr. In order to prove completeness we have to prove:

Proposition 4. *The postulate PA9 is canonically valid, i.e,*

$$R^{C2}abcd \ \& \ d \in S^C \Rightarrow \left(\exists x, y \in K^C\right)\left(\exists z \in S^C\right)\left[R^C acx \ \& \ R^C bcy \ \& \ R^C yxz\right]$$

This proposition follows immediately from

LEMMA 5. *Let a, b, $c \in K^T$, d a consistent member in K^T and $R^{T2}abcd$. Then, there are x, $y \in K^T$ and $z \in S^C$ such that R^Tacx, R^Tbcy and R^Tyxz.*

PROOF. The proof is similar to that of lemma 1, use A9g($n = 1$), T5 and T16. □

7. Two final notes on Bpcr

First, we note that although Bpcr is included in the logic of Relevance R, it is not included in e.g, the logic of Entailment E or Lewis's modal logic S5: if

A8. $B \to [(A \to \neg B) \to \neg A]$

or

T3. $\neg B \to [(A \to B) \to \neg A]$

are added to E, the resulting logic is the logic of Relevance R; if A8 (or T3) is added to Lewis's S3, the result is classical propositionsl logic (proof of this fact is left to the reader).

Secondly, we have remarked that versions of *assertion*

$$A \to [(A \to C) \to C]$$

permutation

$$[A \to (B \to C)] \to [B \to (A \to C)]$$

conditioned modus ponens

$$B \to [[A \to (B \to C)] \to (A \to C)]$$

contraction

$$[A \to (A \to C)] \to (A \to C)$$

self-distribution of the conditional

$$[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$$

permuted self-distribution of the conditional

$$(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$$

and *importation*

$$[A \to (B \to C)] \to [(A \wedge B) \to C]$$

restricted to the case in which C is a negative formula are provable in Bpcr.

We now prove that the unrestricted versions just mentioned are unprovable in Bpcr.

Consider the three following sets of matrices (in (I), 1 and 2 are the designated values; in (II), (III), 2 is the only designated value):

(I)

$\to$	0	1	2	$\neg$
0	1	1	2	2
1	0	1	2	2
2	0	0	2	2

(II)

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	0	2	2	0
2	0	0	2	0

(III)

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	2
2	0	1	2	2

In the three cases, the matrices for $\wedge$ and $\vee$ are the following:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

PROOF. Each one of these three sets satisfies the axioms are rules of Bpcr but:

(I) does not satisfy the suffixing axiom $(A \to B) \to [(B \to C) \to (A \to C)]$ (for example, $v(A) = v = (B) = 2$, $v(C) = 1$).

(II) does not satisfy:
$A \to [(A \to B) \to B]$
$(v(A) = v(B) = 1)$
$[A \to (B \to C)] \to [B \to (A \to C)]$
$(v(A) = 2,\ v(B) = v(C) = 1)$
$B \to [[A \to (B \to C)] \to (A \to C)]$
$(v(A) = 2,\ v(B) = v(C) = 1)$
(III) does not satisfy:
$[A \to (A \to B)] \to (A \to B)$
$(v(A) = 1,\ v(B) = 0)$
$[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

$(v(A) = v(B) = 1,\ v(C) = 0)$
$(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$
$(v(A) = v(B) = 1,\ v(C) = 0)$
$[A \to (B \to C)] \to [(A \land B) \to C]$
$(v(A) = v(B) = 1,\ v(C) = 0)$

□

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