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A Routley-Meyer semantics for Gödel 3-valued logic and its paraconsistent counterpart

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Abstract. Routley-Meyer semantics (RM-semantics) is defined for Gödel 3-valued logic G3 and some logics related to it among which a paraconsistent one differing only from G3 in the interpretation of negation is to be remarked. The logics are defined in the Hilbert-style way and also by means of proof-theoretical and semantical consequence relations. The RM-sematics is defined upon the models for Routley and Meyer's basic positive logic B_+, the weakest positive RM-semantics. In this way, it is to be expected that the models defined can be adapted to other related many-valued logics.

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1. Introduction

The aim of this paper is to provide a Routley-Meyer semantics (RM-semantics) for Gödel 3-valued logic G3 and a 3-valued paraconsistent logic related to it.

Gödel logics were introduced in [6] as a way of showing that propositional intuitionistic logic does not have a finite characteristic matrix. In [5], Dummett proved that if the linearity axiom $(A \to B) \vee (B \to A)$ is added to propositional intuitionistic logic, the resulting system (LC) has an infinite-valued characteristic matrix (see, e.g., [2] on Gödel logics). Now, before defining G3 and its paraconsistent counterpart, we shall specify the logical language used in the paper.

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Remark 1.1 (Languages, logics). *The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$ and some or all of the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, C, etc. (possibly with subscripts $0, 1, ..., n$) are metalinguistic variables. We shall consider, from the proof-theoretical point of view, propositional logics formulated in the Hilbert-style way, that is, logics axiomatized by means of a finite set of axioms (actually, axiom schemes) and a finite set of rules of derivation. Then, we shall define (proof-theoretical) consequence relations on these Hilbert-style formulations in order to be able to derive consequences from sets of premises, which are not necessarily theorems. We shall refer by $\mathcal{P}$ and $\mathcal{F}$ to the set of all propositional variables and the set all wffs, respectively.*

Definition 1.2 (3-valued matrices). *Let S_3 be the set $\{0, \frac{1}{2}, 1\}$ where $0 \leq \frac{1}{2} \leq 1$ and 1 is the only designated value. Then:*

(1) The 3-valued matrix MG3 is defined by the following truth tables:

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

	$\neg$
0	1
$\frac{1}{2}$	0
1	0

(2) The 3-valued matrix MG3$_{\mathrm{L}}$ is defined exactly as MG3, except for the table for negation, which is the following:

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

Given MG3 and MG3$_{\mathrm{L}}$ interpretations and validity are as follows.

Definition 1.3 (Interpretations, validity). *An MG3-interpretation (MG3$_L$-interpretation), I, is a function from $\mathcal{F}$ to S_3 according to the truth tables in MG3 (MG3$_L$). Then, a wff A is MG3-valid (MG3$_L$-valid) (in symbols, $\vDash_{MG3} A$ / $\vDash_{MG3_L} A$) iff $I(A) = 1$ for all MG3-interpretations (MG3$_L$-interpretations) I. A rule of derivation $A_1, ..., A_n \Rightarrow B$ preserves MG3-validity (MG3$_L$-validity) iff B is MG3-valid (MG3$_L$-valid) if each A_i $(1 \leq i \leq n)$ is MG3-valid (MG3$_L$-valid).*

Remark 1.4 (On the label MG3$_{\mathbf{L}}$). *By the subscript L, we intend to remark that negation is defined as in Lukasiewicz many-valued logics, i.e., $\neg x = 1-x$. Notice, by the way, that this type of negation could be added to the positive* *fragment of each one of the Gödel logics G4, G5,..., Gn,...,G∞ (we shall briefly return to this question in the last section of the paper —Remark 6.25).*

Now, it is possible to define on the 3-valued matrices (as on any many-valued matrix) two different types of consequence relations.

Firstly, MG3-interpretations and $\mathrm{MG3_L}$-interpretations of sets of wffs are defined.

Definition 1.5 (MG3- ($\mathbf{MG3_L}$)-interpretations of sets of wffs). *Let I be an arbitrary MG3-interpretation ($MG3_L$-interpretation). Then, for any set of wffs Γ, $I(\Gamma) = \inf\{I(A) : A \in \Gamma\}$.*

Then, we set

Definition 1.6 (Truth-preserving consequence relation). *Let Γ be a set of wffs and A be a wff. Then, $\Gamma \vDash^1_{MG3} A$ ($\Gamma \vDash^1_{MG3_L} A$) iff if $I(\Gamma) = 1$, then $I(A) = 1$ for each MG3-interpretation ($MG3_L$-interpretation) I.*

Definition 1.7 (Degree of truth-preserving consequence relation). *Let Γ be a set of wffs and A be a wff. Then, $\Gamma \vDash^{<}_{MG3} A$ ($\Gamma \vDash^{<}_{MG3_L} A$) iff $I(\Gamma) \leq I(A)$ for each MG3-interpretation ($MG3_L$-interpretation) I.*

These two ways of understanding the notion of semantical consequence in many-valued logics are not in general equivalent. Actually, we have:

Proposition 1.8 (On the 1- and <-consequence relations). *Let Γ be any set of wffs and A a wff. Then, (1) $\Gamma \vDash^1_{MG3} A$ iff $\Gamma \vDash^{<}_{MG3} A$; (2) if $\Gamma \vDash^{<}_{MG3_L} A$ then $\Gamma \vDash^1_{MG3_L} A$ (the converse, however, does not hold).*

Proof. (1) Cf. e.g., [2] (Proposition 2.15); (2) cf. [10] (Proposition 1.16). □

We shall refer by symbols $\vDash^1_{\mathrm{MG3}}$, $\vDash^1_{\mathrm{MG3_L}}$, $\vDash^{<}_{\mathrm{MG3}}$ and $\vDash^{<}_{\mathrm{MG3_L}}$ to the truth-preserving and degree of truth-preserving consequence relations defined in Definition 1.6 and Definition 1.7, respectively.

Then, the logics treated in this paper are the following:

1. G3: The set of all MG3-valid formulas.
2. $\mathrm{G3_L}$: The set of all $\mathrm{MG3_L}$-valid formulas.
3. $\mathrm{G3^1}$ (or, equivalently, $\mathrm{G3^<}$): The logic determined by the relation $\vDash^1_{\mathrm{MG3}}$ (equivalently, $\vDash^{<}_{\mathrm{MG3}}$).
4. $\mathrm{G3^1_L}$: The logic determined by the relation $\vDash^1_{\mathrm{MG3_L}}$.
5. $\mathrm{G3^<_L}$: The logic determined by the relation $\vDash^{<}_{\mathrm{MG3_L}}$.

It is understood that a logic S is determined by the relation $\vDash$ if S is sound and complete w.r.t. $\vDash$.

Now, in [10] it is proved that G3 and $\mathrm{G3_L}$ can be axiomatized as extensions of Routley and Meyer's basic positive logic $\mathrm{B_+}$: $\mathrm{G3_{(B_+)}}$ and $\mathrm{G3_{L(B_+)}}$, respectively (see the appendix). And, on the other hand, it is shown in the same paper that $G3^1$, $G3^1_L$ and $G3^{<}_L$ can be axiomatized by three proof-theoretical relations: $\vdash^1_{G3}$, $\vdash^1_{G3_L}$ and $\vdash^{<}_{G3_L}$, respectively, that are coextensive with $\vDash^1_{MG3}$, $\vDash^1_{MG3_L}$ and $\vDash^{<}_{MG3_L}$, respectively. The consequence relation $\vdash^1_{G3}$ is built upon $G3_{(B_+)}$ and the relations $\vdash^1_{G3_L}$ and $\vdash^{<}_{G3_L}$, upon $G3_{L(B_+)}$ (see Section 5 and Section 6).

As it is well-known, the RM-semantics were introduced in the early seventies of the last century for interpreting relevant logics (cf. [11] and references therein). But this semantics can be used for interpreting logics in general provided these logics contain the basic positive logic B_+. Therefore, it seemed worth the while to try and provide an RM-semantics for the five logics listed above. And this is in particular the aim of this paper. We think that this result has some interest from the Universal Logic enterprise perspective in the sense that it connects Gödel many-valued logics and similar systems as, for instance, Łukasiewicz logics, to relevant logics from the point of view of the latter, the 3-termed relational point of view, in particular. Our RM-models are of some interest since they are not built upon some strong positive relevant logic, but on the minimal positive one in the RM-semantics. Consequently, it is to be expected that they can be adapted, as suggested above, to other many-valued logics.

The structure of the paper is as follows. (At the end of it, an appendix is added where the logics in the paper along with some of the theorems of G3 and $G3_L$ are listed.) In Section 2, we prove some facts about the different classes of theories built upon the three basic logics in the paper: BM′, G3 and $G3_L$. The labels G3 and $G3_L$ are abbreviations for $G3_{(B_+)}$ and $G3_{L(B_+)}$. These abbreviations are justified since, as remarked above, $G3_{(B_+)}$ and $G3_{L(B_+)}$ axiomatize MG3 and $MG3_L$, respectively. The logic BM′ is an extension of Sylvan and Plumwood's minimal logic BM (see the appendix). The results in this section are used in the completeness proofs in the paper. In Section 3, we provide an RM-semantics for the logic BM′. Leaning on this semantics, in Section 4, RM-semantics are defined for G3 and $G3_L$. In Section 5, an RM-semantics is presented w.r.t. which $G3^1$ is sound and complete. These results can be considered as the "strong" soundness and completeness of G3. The development of $G3^1$ sets the pattern for the treatment of $G3^1_L$ and $G3^{<}_L$ in the following section, Section 6. As it was pointed out above, $G3^1_L$ and $G3^{<}_L$ are axiomatized by defining two different proof-theoretical relations, $\vdash^1_{G3_L}$ and $\vdash^{<}_{G3_L}$ on $G3_{L(B_+)}$. Now, $G3^{<}_L$ is paraconsistent, but $G3^1_L$ is not. Thus, $G3^{<}_L$ is the "paraconsistent counterpart" we propose to G3 or $G3^1$. The paper is ended with the proof that $G3^{<}_L$ is actually paraconsistent together with some remarks on the question.

2. Theories, primeness, consistency

In this section we shall prove some facts about different classes of theories built upon the logics BM′, G3 and $G3_L$ (see the definition of these logics and some theorems and rules of BM′, G3 and $G3_L$ to be used throughout the paper in the appendix). These results are used in the completeness proof of the following sections. We begin by defining the notion of a theory.

2.1. Theories and classes of theories

Definition 2.1 (Theories). *Let S be a logic defined on a propositional language with at least the connectives $\rightarrow$ and $\wedge$ (cf. Remark 1.1). An S-theory is a set of formulas closed under Adjunction (Adj) and provable S-implication (S-imp). That is, a is an S-theory if whenever A, $B \in a$, then $A \wedge B \in a$; and whenever $A \rightarrow B$ is a theorem of S and $A \in a$, then $B \in a$.*

The following definition classifies S-theories into different special classes.

Definition 2.2 (Classes of theories). *Let a be an S-theory. We set: (1) a is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; (2) a is regular iff all theorems of S belong to it; (3) a is empty iff no wff belongs to it; (4) a is trivial iff every wff belongs to it; (5) a is w-inconsistent (inconsistent in a weak sense) iff for some theorem A of S, $\neg A \in a$. Then, a is w-consistent (consistent in a weak sense) iff a is not w-inconsistent (cf. [9] on the label "w-consistent"); (6) a is sc-inconsistent (inconsistent according to the standard concept) iff for some wff $A, A \wedge \neg A \in a$. Then, a is sc-consistent (consistent according to the standard concept) iff it is not sc-inconsistent; (7) a is complete iff for every wff A, $A \in a$ or $\neg A \in a$.*

Thus, for example, a BM′-theory is a set of formulas closed under Adj and BM′-imp. And a G3-theory is w-consistent if it does not contain the negation of a theorem of G3, etc. On the other hand, in what follows, we will use the term "theory" in general when referring to properties of theories predicable of BM′-theories, G3-theories and $G3_L$-theories. Otherwise, it will be specified to which type of theories the properties in question are predicable of.

2.2. Consistency, regularity, non-emptiness, non-triviality

Firstly, we prove a couple of easy but useful propositions on theories in the general sense just pointed out. (The rules Veq and Efq used below are defined in the appendix.)

Proposition 2.3 (Regularity and non-emptiness). *Any theory a is regular iff it is non-empty.*

Proof. Immediate by the rule Veq. □

Proposition 2.4 (W-consistency and non-triviality). *Any theory a is w-consistent iff it is not trivial.*

Proof. Immediate by the rule Efq. □

Concerning G3 and $G3_L$, we have:

Proposition 2.5 (Closure of G3-($\mathbf{G3_L}$-)theories under MP and Veq). *Let a be a G3-theory or a $G3_L$-theory. Then, a is closed under MP and Veq. That is, for any $A, B \in \mathcal{F}$, (1) if $A \to B \in a$ and $A \in a$, then $B \in a$; (2) if $A \in a$, then $B \to A \in a$.*

Proof. Immediate by T1 and A11. □

Concerning G3, it is proved:

Proposition 2.6 (G3-theories are closed under Efq and Ecq). *Any G3-theory is closed under Efq and Ecq. That is, for any $A, B \in \mathcal{F}$, (1) if $A \in a$, then $\neg A \to B \in a$; (2) if $A \wedge \neg A \in a$, then $B \in a$.*

Proof. 1 and 2 are immediate by T6 and T7, respectively. □

Proposition 2.7 (Sc-consistenty and w-consistency in G3-theories). *Let a be a G3-theory. Then, a is sc-consistent iff a is w-consistent iff a is not trivial.*

Proof. As G3-theories are closed under Ecq, it is clear that any G3-theory is sc-consistent iff it is non-trivial. Then, Proposition 2.7 follows by Proposition 2.4. □

Remark 2.8 (Sc-consistency, w-consistency and regularity). *Regular, prime, complete and w-consistent $G3_L$-theories are not in general sc-consistent. But, on the other hand, it is clear that any regular, sc-consistent $G3_L$-theory is w-consistent. (We shall return to this question in the last section of the paper.)*

2.3. Extensions to prime theories

Next, we prove the primeness lemmas.

Lemma 2.9 (Extension to prime, w-consistent theories). *Let a be a non-empty theory and A a wff such that $A \notin a$. Then, there is a regular, prime, w-consistent theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Assume the hypothesis of Lemma 2.9. Firstly, notice that a is a regular and w-consistent theory (Proposition 2.3 and Proposition 2.4). Then, extend a to a maximal theory x without A. It is clear that x is regular and w-consistent. Finally, it is not difficult to prove that x is prime (cf. [11], Chapter 4, where it is shown how to proceed in the case of an ample class of logic including B_+). □

In the case of G3, Lemma 2.9 can be strengthen to:

Lemma 2.10 (Extension to prime, sc-consistent G3-theories). *Let a be a non-empty G3-theory and A a wff such that $A \notin a$. Then, there is a regular, prime, sc-consistent theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Immediate by Proposition 2.7 and Lemma 2.9. □

2.4. *-images of prime theories

In what follows, we shall use the Routley operator * (cf. [11] and references therein) in order to define, as in relevant logics, the *-images of prime theories.

Definition 2.11 (*-images of prime theories). *Let a be a prime theory. The set a^* is defined as follows: $a^* = \{A : \neg A \notin a\}$.*

Next, we prove a couple of lemmas on the relationship between prime theories and their *-images.

Lemma 2.12 (Primeness of *-images). *Let a be a prime theory. Then, a^* is a prime theory as well.*

Proof. (cf. [11]) Let a be a prime theory. (1) a^* is closed under BM′-(G3–, $\mathrm{G3_L}$-)imp, by Con. (2) a^* is closed under Adj, by $\neg(A \wedge B) \to (\neg A \vee \neg B)$ (A8). (3) a^* is prime, by $(\neg A \wedge \neg B) \to \neg(A \vee B)$ (A7). □

Lemma 2.13 (*-images and negation). *(1) Let a be a prime BM′- (G3-) theory. For any $A \in \mathcal{F}$, if $A \in a$, then $\neg A \notin a^*$. (2) Let a be a prime $G3_L$-theory. For any $A \in \mathcal{F}$, $A \in a$ iff $\neg A \notin a^*$.*

Proof. Case 1 follows by A9 ($A \to \neg\neg A$) and Definition 2.11. Case 2 (from right to left) follows from A17 ($\neg\neg A \to A$) and Definition 2.11; the inverse direction follows by case 1. □

In the sequel we prove some facts about the relationship between consistency, regularity and completeness in prime theories and their images. But in order to do this, we have to distinguish between theories in general and $\mathrm{G3_L}$-theories in particular. (Cf. [10].)

Lemma 2.14 (Consistency, regularity, completeness and *-images I). *Let a be a prime theory. Then, (1) a is w-consistent iff a^* is regular; (2) a is sc-consistent iff a^* is complete; (3) If a is regular, then a^* is w-consistent; (4) If a is complete then a^* is sc-consistent.*

Proof. (Cf. Definition 2.2). Case 1, and case 2 (from left to right) are immediate by Definition 2.2 and Definition 2.11. Case 2 (right to left): let a^* be complete. Suppose for reductio that a is sc-inconsistent. Then there is some wff A such that $A \in a$ and $\neg A \in a$. By Lemma 2.13(1) and Definition 2.11, $A \notin a^*$ and $\neg A \notin a^*$, contradicting the completeness of α^*. Now, cases 3 and 4 are proved similarly. □

Lemma 2.15 (Consistency, regularity, completeness and *-images II). *Let a be a prime $G3_L$-theory. Then, (1) a is regular iff a^* is w-consistent; (2) a is complete iff a^* is sc-consistent.*

Proof. 1 and 2 from left to right follow by Lemma 2.14(3) and 2.14(4), respectively. The converses of 1 and 2 are proved similarly as cases 3 and 4 in Lemma 2.14 except that now Lemma 2.13(2) is used. □

In the sequel we prove some facts that will turn out to be essential in order to prove the completeness of the logics $\mathrm{G3}^1_\mathrm{L}$ and $\mathrm{G3}^<_\mathrm{L}$, in the last section of the paper.

Proposition 2.16 (Sc-consistency or completeness). *Let a be a prime $G3_L$-theory. If a is sc-inconsistent, then a is complete.*

Proof. Immediate by Definition 2.1, Definition 2.2 and A15. (Notice that if a is an sc-inconsistent G3-theory, then a is trivial by Proposition 2.7.) □

Next, it is proved that for any non-empty $\mathrm{G3_L}$-theory a lacking a given formula, there is a prime, sc-consistent theory x without the same formula (this $\mathrm{G3_L}$-theory x does not necessarily include a).

Lemma 2.17 (From w-consistent to sc-consistent $\mathbf{G3_L}$-theories). *Let a be a non-empty $G3_L$-theory such that for some wff A, $A \notin a$. Then, there is a prime, regular, sc-consistent $G3_L$-theory x such that $A \notin x$.*

Proof. Assume the hypothesis of Lemma 2.17. By Lemma 2.9, there is a prime, regular, w-consistent $\mathrm{G3_L}$-theory y such that $a \subseteq y$ and $A \notin y$. Suppose that y is sc-inconsistent. Then, y is complete. Moreover, $\neg A \in y^*$ since $A \notin y$ (by Lemma 2.13(2)). On the other hand, y^* is prime (Lemma 2.12), regular (Lemma 2.14(1) and sc-consistent (Lemma 2.14(4)). So, $A \notin y^*$ ($\neg A \in y^*$). Therefore, either y or y^* is the required x in the statement of Lemma 2.17. (Notice that a is not necessarily included in y^*). □

Finally, we prove that prime, sc-consistent $\mathrm{G3_L}$-theories are closed under the rule Con. (It is clear that G3-theories are closed under Con: by A12 and A9 $(A \to B) \to (\neg B \to \neg A)$ is an immediate theorem.)

Proposition 2.18 (When $\mathbf{G3_L}$theories are closed under Con). *Let a be a $G3_L$-theory. Then, if a is prime and sc-consistent, a is closed under Con. That is, for any A, $B \in \mathcal{F}$, if $A \to B \in a$, then $\neg B \to \neg A \in a$.*

Proof. Suppose for arbitrary wffs A, B, (1) $A \to B \in a$. Suppose further (2) $\neg B \notin a$. By T4, (3) $(A \to B) \to [\neg B \vee (\neg B \to \neg A)]$. So, by 1 and 3, $\neg B \vee (\neg B \to \neg A) \in a$ whence, by 2 and primeness, $\neg B \to \neg A \in a$. Suppose, on the other hand, (4) $\neg B \in a$. By T5, (5) $(A \to B) \to [B \vee (\neg B \to \neg A)]$. By 1 and 5, (6) $B \vee (\neg B \to \neg A) \in a$. But, given 4, by sc-consistency, $B \notin a$, whence by 6 and primeness, $\neg B \to \neg A \in a$. Therefore, a is closed under Con. $\Box$

3. A Routley-Meyer semantics for BM′

We recall that by the label BM′ we refer to the logic $\mathrm{BM}_{\mathrm{dn,v,f}}$, that is, to the result of adding the rules Veq, Efq and the axiom of double negation $A \to \neg\neg A$ to Sylvan and Plumwood's minimal logic BM. (See the appendix.) In this section, we will define an RM-semantics for BM′ as a preliminary step for providing an RM-semantics for G3 and $\mathrm{G3_L}$.

3.1. BM′-models. Soundness of BM′

Definition 3.1 (BM′-models). *A BM′-model is a structure* $(K, R, ^*, \vDash)$ *where* K *is a set,* R *is a ternary relation on* K *and* * *a unary operation on* K *subject to the following definitions and postulates for all* a, b, $c \in K$*:*

d1. $a \leq b =_{df} (\exists x \in K)Rxab$

P1. $a \leq a$

P2. $(a \leq b \ \& \ Rbcd) \Rightarrow Racd$

P3. $a \leq b \Rightarrow b^* \leq a^*$

P4. $a \leq a^{**}$

Finally, $\vDash$ *is a relation from* K *to* $\mathcal{F}$ *such that the following conditions are satisfied for all* $p \in \mathcal{P}$*,* A*,* $B \in \mathcal{F}$ *and* $a \in K$*:*

(i) $(a \leq b \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ *iff* $a \vDash A$ *and* $a \vDash B$

(iii) $a \vDash A \vee B$ *iff* $a \vDash A$ *or* $a \vDash B$

(iv) $a \vDash A \to B$ *iff for all* $b, c \in K$*,* $(Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

(v) $a \vDash \neg A$ *iff* $a^* \nvDash A$

Definition 3.2 (BM′-validity). *A formula* A *is BM′-valid (in symbols,* $\vDash_{BM'} A$*) iff* $a \vDash A$ *for all* $a \in K$ *in all BM′-models.*

Remark 3.3 (BM′-models and relevant models). *The only (but crucial) difference between BM′-models and standard models for relevant logics is the following. In the latter, a distinguished subset of* K*,* O*, is included. It is w.r.t. this set that the relation* $\leq$ *and, most of all, validity, are defined as follows:* $a \leq b =_{df} (\exists x \in O)Rxab$*;* A *is valid iff* $a \vDash A$ *for all* $a \in O$ *in all models (cf. [11]). Now, let us drop the postulate P4 from BM′-models. Then, BM-models (models for Sylvan and Plumwood's BM) and BM′-models are indistinguishable from each other save for the point just remarked (cf. [3], Chapter 6). Nevertheless, this set* O *will be introduced in the models in the* *last section of the paper (Section 6) in order to define an RM-semantics for* $G3^1_L$.

Before proving soundness we note a couple of useful propositions.

Proposition 3.4 (Hereditary condition). *For any BM′-model,* $a, b \in K$ *and* $A \in \mathcal{F}$, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.

Proof. Induction on the length of A. The conditional case is proved with P2, and the negation case with P3. □

Lemma 3.5 (Entailment lemma). *For* $A, B \in \mathcal{F}$, $\vDash_{BM'} A \to B$ *iff* $(a \vDash A \Rightarrow a \vDash B$, *for all* $a \in K$ *in all BM′-models).*

Proof. By P1 and Proposition 3.4. □

Theorem 3.6 (Soundness of BM′). *For each* $A \in \mathcal{F}$, *if* $\vdash_{BM'} A$, *then* $\vDash_{BM'} A$.

Proof. The validity of the axioms and rules of B_+ and that of A7, A8 and Con is proved similarly as in the standard semantics (see, e.g., [11]). Then, the validity of Veq and Efq is immediate by Definition 3.1, Definition 3.2 and Lemma 3.5. □

3.2. Completeness of BM′

In what follows, we proceed into the completeness proof. We begin by defining the canonical model. As in the preceding section, in what follows, by the term "theory" we will refer to any arbitrary BM′-theory, G3-theory or $G3_L$-theory (cf. Definition 2.1, Definition 2.2).

Definition 3.7 (Canonical models). *Let* K^T *be the set of all theories (cf. Definition 2.1) and* R^T *be defined on* K^T *as follows: for all* $a, b, c \in K^T$ *and* $A, B \in \mathcal{F}$, $R^T abc$ *iff* $(A \to B \in a \ \& \ A \in b) \Rightarrow B \in c$. *Now, let* K^C *be the set of all non-empty, prime, w-consistent theories. On the other hand, let* R^C *be the restriction of* R^T *to* K^C, *and* ^{*C} *be defined on* K^C *as follows: for each* $a \in K^C$, $a^* = \{A : \neg A \notin a\}$ *(cf. Definition 2.11). Finally,* $\vDash^C$ *is defined as follows: for any* $a \in K^C$ *and* $A \in \mathcal{F}$, $a \vDash^C A$ *iff* $A \in a$. *Then, the canonical model is the structure* $(K^C, R^C, {}^{*C}, \vDash^C)$.

Definition 3.8 (The canonical BM′-model). *The canonical BM′-model is the structure* $(K^C, R^C, {}^{*C}, \vDash^C)$ *where* K^C *is the set of all (non-empty, prime, w-consistent) BM′-theories.*

Now, in order to prove that the canonical model is in fact a model, we need to prove some preliminary facts.

Lemma 3.9 (Defining x for a, b in R^T). *Let* a, b *be non-empty elements in* K^T. *The set* $x = \{B : \exists A[A \to B \in a \ \& \ A \in b]\}$ *is a non-empty theory such that* $R^T abx$.

Proof. It is easy to prove that x is a theory. Moreover, x is non-empty: let $A \in b$; by A1 and Proposition 2.3, $A \to A \in a$. So, $A \in x$. Finally, $R^T abx$ is immediate by definition of R^T. □

Lemma 3.10 (R^T and w-consistency). *Let a, b be non-empty elements in K^T and c a w-consistent member in K^T such that $R^T abc$. Then, a and b are w-consistent.*

Proof. (1) Let A be a theorem. Suppose a is w-inconsistent and let $B \in b$. As a is trivial (Proposition 2.4), $B \to \neg A \in a$. So $\neg A \in c$ contradicting the w-consistency of c. (2) Suppose b is w-inconsistent. Then, $\neg A \in b$ for some theorem A. Now, $\neg A \to \neg A \in a$ (Proposition 2.3). So, $\neg A \in c$, contradicting the w-consistency of c. □

Lemma 3.11 (^{*C} is an operation on K^C). *Let $a \in K^C$. Then, $a^* \in K^C$. That is, ^{*C} is an operation on K^C.*

Proof. Let $a \in K^C$. By Lemma 2.12, a^* is prime; by Lemma 2.14(1), a^* is regular. Finally, by Lemma 2.14(3), a^* is w-consistent. □

Lemma 3.12 (Extending a and b in $R^T abc$ to prime theories). *Let a, b be non-empty elements in K^T and $c \in K^C$ such that $R^T abc$. Then, there are x, $y \in K^C$ such that $a \subseteq x$, $b \subseteq y$, $R^T xbc$ and $R^T ayc$.*

Proof. Given the hypothesis of Lemma 3.12, we build up prime non-null theories x, y such that $R^T xbc$ and $R^T ayc$ (cf. [11]). By Lemma 3.10, x and y are, in addition, w-consistent. □

Lemma 3.13 ($\leq^C$ and $\subseteq$ are coextensive). *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. Let S refer to any of the logics BM′, G3 or G3_L. From left to right it is immediate. So, suppose $a \subseteq b$ for a, $b \in K^C$. Clearly $R^T \text{S}aa$ (cf. Definition 2.1). Then, by Lemma 3.12, there is some x in K^C such that $\text{S} \subseteq x$ and $R^C xaa$. By hypothesis, $R^C xab$ i.e., $a \leq^C b$ (cf. d1 in Definition 3.1). □

Notice that lemmas 3.9-3.13 hold in any of the canonical models defined in Definition 3.7.

Lemma 3.14 (The canonical BM′-model is a BM′-model). *Let $(K^C, R^C, *^C, \vDash^C)$ be the canonical BM′-model. Then, it is indeed a BM′-model .*

Proof. *R^C is clearly a ternary relation on K^C and $*^C$ is an operation on* K^C (Lemma 3.11). So, we have to prove the following facts.

1. *The set K^C is not empty.*
2. Clauses (i)-(v) in Definition 3.1 are satisfied by the canonical BM′-model.
3. Postulates P1-P4 hold in the canonical BM′-model.

Now, we shall prove facts 1-3 for any of the canonical models considered in Definition 3.7. Actually, we prove that facts 1-3 hold in the canonical model of no matter which extension of BM′ with the same propositional language. But, for definiteness, we shall refer by S to any of the logics BM′, G3 or $\mathrm{G3_L}$.

Fact 1 is immediate by Lemma 2.9: the logic S is a non-null w-consistent theory.

Fact 2: clause (i) is immediate by Lemma 3.13. Clauses (ii), (iii), (v) and clause (iv) from left to right are proved as in the semantics for E or R (see, e.g., [11]). So, let us prove clause (iv) from right to left. Suppose for A, $B \in \mathcal{F}$ and $a \in K^C$, $A \rightarrow B \notin a$. We prove that there are x, $y \in K^C$ such that $R^C axy$, $A \in x$ and $B \notin y$.

The sets $z = \{C : \vdash_{\mathrm{S}} A \rightarrow C\}$ and $u = \{C : \exists D[D \rightarrow C \in a \ \& \ D \in z]\}$ are theories such that $R^T azu$. Now, z is w-consistent. Otherwise, $B \in z$ (Proposition 2.4), that is, $\vdash_{\mathrm{S}} A \rightarrow B$, and then, $A \rightarrow B \in a$ (Proposition 2.3), contradicting the hypothesis. Moreover, $A \in z$ (by A1). So, u is non-empty (Lemma 3.9). On the other hand, $B \notin u$ (if $B \in u$, then $A \rightarrow B \in a$ contradicting the hypothesis). Therefore, u is w-consistent (Proposition 2.4). Consequently, we have non-empty, w-consistent theories z, u such that $R^T azu$, $A \in z$ and $B \notin u$. Now, by Lemma 2.9, there is some $y \in K^C$ such that $u \subseteq y$ and $B \notin y$. Obviously, $R^C azy$. Next, by Lemma 3.12, there is some $x \in K^C$ such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have prime, non-empty, w-consistent theories x, y such that $A \in x$ (i.e., $x \vDash^C A$), $B \notin y$ ($y \nvDash^C B$) and $R^C axy$, as was to be proved.

Fact 3: by using Lemma 3.13, P1-P3 are immediate. Then, P4 follows by Definition 2.11 and Lemma 2.13. □

Finally, we prove the completeness theorem after noting a corollary of Lemma 2.9.

Corollary 3.15 (Extending BM′ to a member in K^C). *Suppose $\nvdash_{BM'} A$. Then, there is some $x \in K^C$ such that $A \notin x$.*

Proof. Immediate by Lemma 2.9, since BM′ is, of course, a non-empty theory lacking A. □

Theorem 3.16 (Completeness of BM′). *For any $A \in \mathcal{F}$, if $\vDash_{BM'} A$, then $\vdash_{BM'} A$.*

Proof. Suppose $\nvdash_{\mathrm{BM}'} A$. By Corollary 3.15, there is some $x \in K^C$ such that $A \notin x$. By Definition 3.8 and Lemma 3.14, $x \nvDash^C A$. Therefore $\nvDash_{\mathrm{BM}'} A$ by Definition 3.2. □

4. A Routley-Meyer semantics for G3 and $\mathbf{G3_L}$

Leaning on the RM-semantics for BM′, we provide an RM-semantics for G3 and $\mathrm{G3_L}$. That is, for the logics $\mathrm{G3}_{(\mathrm{B}_+)}$ and $\mathrm{G3}_{\mathrm{L}(\mathrm{B}_+)}$ as axiomatized in the appendix. We begin by defining G3-models, $\mathrm{G3_L}$-models and the respective concepts of validity.

4.1. G3-models, $\mathbf{G3_L}$-models. Soundness of G3 and $\mathbf{G3_L}$

Definition 4.1 (G3-models). *A G3-model is a structure* $(K, R,^*, \vDash)$ *where* K, R, * *and* $\vDash$ *are defined similarly as in BM′-models (cf. Definition 3.1) except that the following definition and postulates hold for all a, b, c, d* $\in K$*: P1, P2, P4 and*

$$d2.\ R^2abcd =_{df} (\exists x \in K)(Rabx \ \ \&\ \ Rxcd)$$

$$P5.\ Rabc \Rightarrow R^2abbc$$

$$P6.\ Rabc \Rightarrow a \leq c$$

$$P7.\ Rabc \Rightarrow Rac^*b^*$$

$$P8.\ Rabc \Rightarrow b \leq a^*$$

$$P9.\ Rabc \Rightarrow a^* \leq c \text{ or } b \leq a$$

Definition 4.2 ($\mathbf{G3_L}$-models). *A* $G3_L$*-model is a structure* $(K, R,^*, \vDash)$ *where* K, R, * *and* $\vDash$ *are defined similarly as in BM′-models (cf. Definition 3.1) except that the following definition and postulates hold for all a, b, c, d* $\in K$*: d2, P1, P2, P3, P4, P5, P6, P9 and*

$$P10.\ a^{**} \leq a$$

$$P11.\ Rabc \Rightarrow b \leq a^* \text{ or } b \leq a$$

$$P12.\ a^* \leq a \text{ or } a \leq a^*$$

Remark 4.3 (G3-($\mathbf{G3_L}$-) models are BM′-models). *Notice that P3 is immediate in any G3-model by P7 and d1. Therefore, P1, P2, P3 and P4 hold in all G3-(*$G3_L$*-) models. Consequently, any G3-(*$G3_L$*-)model is a BM′-model. (It is clear, however, that the converse does not hold.)*

Definition 4.4 (G3-validity). *A formula A is G3-valid (in symbols,* $\vDash_{G3} A$*) iff* $a \vDash A$ *for all* $a \in K$ *in all G3-models.*

Definition 4.5 ($\mathbf{G3_L}$-validity). *A formula A is* $G3_L$*-valid (in symbols,* $\vDash_{G3_L} A$*) iff* $a \vDash A$ *for all* $a \in K$ *in all* $G3_L$*-models.*

Before proving soundness we record a couple of useful propositions.

Lemma 4.6 (Entailment lemma). *For* $A, B \in \mathcal{F}$*, (1)* $\vDash_{G3} A \to B$ *iff (*$a \vDash A \Rightarrow a \vDash B$*, for all* $a \in K$ *in all G3-models); (2)* $\vDash_{G3_L} A \to B$ *iff (*$a \vDash A \Rightarrow a \vDash B$*, for all* $a \in K$ *in all* $G3_L$*-models).*

Proof. Immediate by P1 and Proposition 3.4 (cf. Remark 4.3). □

Proposition 4.7 ($a* \vDash \neg A$ **and** $a \nvDash A$)**.** *For any* $A \in \mathcal{F}$*, (1) for any G3-model and* $a \in K$*, if* $a \nvDash A$ *then* $a* \vDash \neg A$*; (2) for any* $G3_L$*-model and* $a \in K$*,* $a \nvDash A$ *iff* $a* \vDash \neg A$*.*

Proof. By clause (v) in Definition 3.1 and P4, and (in the case of $G3_L$-models) by P10, in addition. □

Theorem 4.8 (Soundness of G3). *For* $A \in \mathcal{F}$*, if* $\vdash_{G3} A$*, then* $\vDash_{G3} A$*.*

Proof. The validity of A1-A10, A12 and that of the rules MP, Adj, Pref and Suf is proved as in the standard semantics for, say, E or R. Then, the validity of A11 and that of A13 is easily shown by P6 and P8, respectively. So, let us prove that $(A \vee \neg B) \vee (A \to B)$ (A14) is G3-valid. (The clauses mentioned below are those in Definition 3.1). Suppose, for reductio, that there are A, $B \in \mathcal{F}$ and $a \in K$ in a G3-model such that $a \nvDash (A \vee \neg B) \vee (A \to B)$. Then (1) $a \nvDash A$, $a \nvDash \neg B$ and $a \nvDash A \to B$. By clause (v), (2) $a^* \vDash B$, and by clause (iv), there are b, $c \in K$ in this model such that (3) $Rabc$, $b \vDash A$, $c \nvDash B$. Now, by P9, $Rabc \Rightarrow a^* \leq c$ or $b \leq a$. So, $a^* \leq c$ or $b \leq a$. Suppose $a^* \leq c$. By Proposition 3.4 and 2, $c \vDash B$, contradicting 3. So, suppose $b \leq a$. By Proposition 3.4 and 3, $a \vDash A$, contradicting 1. Therefore, for all $a \in K$ in all G3-models and A, $B \in \mathcal{F}$, $a \vDash (A \vee \neg B) \vee (A \to B)$, i.e., A14 is G3-valid. □

Theorem 4.9 (Soundness of $\mathbf{G3_L}$). *For* $A \in \mathcal{F}$*, if* $\vdash_{G3_L} A$*, then* $\vDash_{G3_L} A$*.*

Proof. A1-A10, A17 and MP, Adj, Suf, Pref and Con are proved as in the standard semantics; A11 and A14 are proved as in Theorem 4.8. So, let us prove the validity of A15 and A16 (we use Lemma 4.6).

$(A \wedge \neg A) \to (B \vee \neg B)$ *(A15) is* $G3_L$*-valid:*

Suppose, for reductio, that there are A, $B \in \mathcal{F}$ and $a \in K$ in a $G3_L$-model such that $a \vDash A \wedge \neg A$, $a \nvDash B \vee \neg B$. Then (1) $a \vDash A$, $a \vDash \neg A$, $a \nvDash B$ and $a \nvDash \neg B$. So, by Proposition 4.7(2), (2) $a^* \vDash \neg B$. By P12, (3) $a^* \leq a$ or $a \leq a^*$. Suppose $a^* \leq a$. By Proposition 3.4, and 2, we have $a \vDash \neg B$, contradicting 1. Suppose then, $a \leq a^*$. By Proposition 3.4 and 1, $a^* \vDash \neg A$, whence, by Proposition 4.7(2), $a \nvDash A$, which contradicts 1. Therefore, A15 is $G3_L$-valid.

$\neg A \to [A \vee (A \to B)$ *(A16) is* $G3_L$*-valid:*

Suppose, for reductio, that there are A, $B \in \mathcal{F}$ and $a \in K$ in some $G3_L$-model such that (1) $a \vDash \neg A$, $a \nvDash A$, $a \nvDash A \to B$. Then, there are b, $c \in K$ in this model such that (2) $Rabc$, $b \vDash A$, $c \nvDash B$. By 1, (3) $a^* \nvDash A$. By 2 and P11, $b \leq a^*$ or $b \leq a$. Suppose $b \leq a^*$. By Proposition 3.4 and 2, $a^* \vDash A$, which contradicts 3. Suppose, then, $b \leq a$. By Proposition 3.4 and 1, $b \nvDash A$, which contradicts 2. Therefore, A16 is $G3_L$-valid. □

4.2. Completeness of G3 and $\mathbf{G3_L}$

Next, we proceed into the proof of completeness. Firstly, the canonical models are defined.

Definition 4.10 (The canonical G3-model). *The canonical G3-model is the structure* $(K^C, R^C, {}^{*C}, \vDash^C)$ *where* K^C *is the set of all (non-empty, prime, w-consistent) G3-theories (cf. Definition 3.7).*

Definition 4.11 (The canonical $\mathbf{G3_L}$-model). *The canonical* $G3_L$*-model is the structure* $(K^C, R^C, {}^{*C}, \vDash^C)$ *where* K^C *is the set of all (non-empty, prime, w-consistent)* $G3_L$*-theories (cf. Definition 3.7).*

We have to prove that the canonical G3-model ($\mathrm{G3_L}$-model) is indeed a G3-model ($\mathrm{G3_L}$-model). Then, completeness is an easy consequence of Lemma 2.9. Before proceeding into the proof, notice that, as remarked above, lemmas 3.9-3.13 hold for G3 and $\mathrm{G3_L}$; that is, they hold when referring either to G3-theories or $\mathrm{G3_L}$-theories.

Lemma 4.12 (The canonical G3-($\mathbf{G3_L}$-)model is a G3-($\mathbf{G3_L}$-)model). *(1) The canonical G3-model is indeed an G3-model. (2) The canonical* $G3_L$*-model is indeed a* $G3_L$*-model.*

Proof. In order to prove Lemma 4.12, we must show that (1) the set K^C is not empty, which follows by lemma 2.9, as in the proof of Lemma 3.14; (2) clauses (i)-(v) are satisfied in the canonical model (the proof is as in Lemma 3.14); and (3) the postulates hold canonically —P1-P4, P5, P7 and P10 are proved as in the standard semantics (cf., e.g., [11]); then, P6 and P8 are easy by A11 and A13, respectively. So, let us prove P9, P11 and P12. These postulates hold both in the canonical G3-model and in the canonical $\mathrm{G3_L}$-model. We prove that they hold in the latter (the proof that they hold in the former is similar. Actually, notice that P11 and P12 are weak forms of, respectively, P8 and P12′ ($a \leq a^*$) —immediate by P8 and P5′ ($Raaa$), in its turn provable by P5. See Proposition 4.14 below).

P9. $R^C abc \Rightarrow a^* \leq^C c$ or $b \leq^C a$ holds in the canonical $\mathrm{G3_L}$-model:

Suppose, for reductio, that there is a $\mathrm{G3_L}$-model and a, b, $c \in K^C$ such that (1) $R^C abc$ but (2) $a^* \nleq^C c$ and $b \nleq^C a$. Then, for some A, $B \in \mathcal{F}$, we have (cf. Lemma 3.13), (3) $A \in a^*$, $A \notin c$, $B \in b$, $B \notin a$. Then, (4) $\neg A \notin a$. Now, by A14, the following is a theorem, $(B \vee \neg A) \vee (B \to A)$. By the primeness of a, 3 and 4, $B \to A \in a$, whence by 1 and 3, $A \in c$, contradicting 3. Therefore, P9 holds in the canonical $\mathrm{G3_L}$-model.

P11. $R^C abc \Rightarrow b \leq^C a^*$ or $b \leq^C a$ holds in the canonical $\mathrm{G3_L}$-model:

Suppose, for reductio, that there is a $\mathrm{G3_L}$-model and a, b, $c \in K^C$ such that (1) $R^C abc$, but (2) $b \nleq^C a^*$ and $b \nleq^C a$. Then, for some A, $B \in \mathcal{F}$, we have (cf. Lemma 3.13), (3) $A \in b$, $A \notin a^*$, $B \in b$, $B \notin a$. Then, (4) $\neg A \in a$. Now, let C be a theorem of $\mathrm{G3_L}$. By A16, $\neg A \to [A \vee (A \to \neg C)]$. So, by

4, $A \vee (A \to \neg C) \in a$. As a is prime, (5) $A \in a$ or $A \to \neg C \in a$. Suppose (6) $A \in a$. Then, by 4, $A \wedge \neg A \in a$ and a is sc-inconsistent. Therefore, a is complete (Proposition 2.16). So, $A \to B \in a$ or $\neg(A \to B) \in a$. Suppose $A \to B \in a$. By $B \notin a$ in 3 and closure under MP (Proposition 2.5), $A \notin a$, contradicting our hypothesis 6. Suppose then $\neg(A \to B) \in a$. As $B \notin a$ and a is complete, $\neg B \in a$, whence by A16, $B \vee (B \to \neg C) \in a$. By primeness of a and 3 ($B \notin a$), $B \to \neg C \in a$. Then, by 1, 3 ($B \in b$), $\neg C \in c$, contradicting the w-consistency of c. Finally, suppose the second alternative in 5, (7) $A \to \neg C \in a$. By 1, 3 ($A \in b$), $\neg C \in c$, contradicting the w-consistency of c. Consequently, P11 holds in the canonical $\mathrm{G3_L}$-model.

P12. $a^* \leq^C a$ or $a \leq^C a^*$ holds in the canonical $\mathrm{G3_L}$-model:

Suppose, for reductio, that there is a $\mathrm{G3_L}$-model and $a \in K^C$ such that $a^* \nleq a$ and $a \nleq a^*$. Then, for A, $B \in \mathcal{F}$ (cf. Lemma 3.13), (1) $A \in a^*$, $A \notin a$, $B \in a$ and $B \notin a^*$. Next, by 1, (2) $\neg B \in a$. So, by 1 ($B \in a$) and 2, $B \wedge \neg B \in a$. Now, by A15, $(B \wedge \neg B) \to (A \vee \neg A)$. Then, (3) $A \vee \neg A \in a$, whence, by the primeness of a, $A \in a$ or $\neg A \in a$. By 1 ($A \notin a$), $\neg A \in a$. But then, $A \notin a^*$, contradicting $A \in a^*$ in 1.

The proof that P12 holds in the canonical $\mathrm{G3_L}$-model ends the proof of Lemma 4.12. □

Finally, we prove the completeness theorems.

Theorem 4.13 (Completeness of G3 and $\mathbf{G3_L}$). *For any $A \in \mathcal{F}$, (1) if $\vDash_{G3} A$, then $\vdash_{G3} A$; (2) if $\vDash_{G3_L} A$, then $\vdash_{G3_L} A$.*

Proof. Similar to that of Theorem 3.16 by leaning on the appropriate corollaries of Lemma 2.9 defined similarly as Corollary 3.15. □

4.3. "Disjunctive Contraposition" is an admissible rule in $\mathbf{G3_L}$

We end this section showing that the rule Disjunctive Contraposition (Dcon) is an admissible rule of $\mathrm{G3_L}$. (This fact is essential in the completeness proof of $\mathrm{G3_L^1}$ in Section 6.) But before we need an auxiliary proposition.

Proposition 4.14 (The postulate P5′). *Let $(K, R,^*, \vDash)$ be a $G3_L$-model. Then, for any $a \in K$, we have, (P5′) Raaa.*

Proof. Assume the hypothesis of Proposition 4.14. By P1, for some $x \in K$, (1) $Rxaa$. By P5, (2) $Rxaa \Rightarrow R^2xaaa$. By 1 and 2, (3) R^2xaaa, whence by d2, for some $y \in K$, (4) $Rxay$ and $Ryaa$. By 4 ($Rxay$) and d1, (5) $a \leq y$. By 4 ($Ryaa$), 5 and P2, (6) $Raaa$, as was to be proved. □

Firstly we show that Dcon preserves $\mathrm{G3_L}$-validity.

Proposition 4.15 (Dcon preserves $\mathbf{G3_L}$-validity). *The rule Dcon, i.e., for A, B, $C \in \mathcal{F}$, (Dcon) $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$ preserves $G3_L$-validity. That is, if $C \vee (A \to B)$ is $G3_L$-valid, then $C \vee (\neg B \to \neg A)$ is $G3_L$-valid.*

Proof. Suppose that there are wffs A, B, C such that $\vDash_{G3_L} C \vee (A \to B)$ but $\nvDash_{G3_L} C \vee (\neg B \to \neg A)$. Then, there is some G3$_L$-model and $a \in K$ such that (1) $a \nvDash C$, $a \nvDash \neg B \to \neg A$. Then, there are b, $c \in K^C$ in this G3$_L$-model such that (2) $Rabc$, $b \vDash \neg B$, $c \nvDash \neg A$, whence (3) $c^* \vDash A$. On the other hand, by 2 ($Rabc$) and P6, (4) $a \leq c$. By 2 ($Rabc$) and d1, (5) $b \leq c$, and, by 4 and P3, (6) $c^* \leq a^*$. Now, by 2 ($Rabc$) and P9, (7) $a^* \leq c$ or $b \leq a$. Suppose (8) $a^* \leq c$. By P3 and P10, (9) $c^* \leq a$. By 1 ($a \nvDash C$), 9 and Proposition 3.4, (10) $c^* \nvDash C$. Now, as $C \vee (A \to B)$ is G3$_L$-valid, by 10, (11) $c^* \vDash A \to B$. On the other hand, by Proposition 4.14, (12) $Rc^*c^*c^*$. So, by 3, 11 and 12, (13) $c^* \vDash B$, whence (14) $c \nvDash \neg B$ and by 5, (15) $b \nvDash \neg B$, contradicting $b \vDash \neg B$ in 2. So, suppose the second alternative in 7, (16) $b \leq a$. By 16 and P3, (17) $a^* \leq b^*$. Now, given P12, we have two possibilities: (18) $a \leq a^*$ or $a^* \leq a$. Suppose (19) $a \leq a^*$. By 1 ($a \nvDash C$) and the G3$_L$-validity of $C \vee (A \to B)$, (20) $a \vDash A \to B$. By 19, 20 and Proposition 3.4, (21) $a^* \vDash A \to B$. Then, by Proposition 4.14, (22) $Ra^*a^*a^*$, and by 3, 6 and Proposition 3.4, (23) $a^* \vDash A$. So, by 21, 22 and 23, (24) $a^* \vDash B$. Then, by 17, 24 and Proposition 3.4, (25) $b^* \vDash B$, whence (26) $b \nvDash \neg B$, contradicting $b \vDash \neg B$ in 2. So, suppose the second alternative in 18: (27) $a^* \leq a$. By 1 ($a \nvDash C$) and 27, (28) $a^* \nvDash C$, whence by 6, (29) $c^* \nvDash A$, contradicting 3.

Consequently, if $\vDash_{G3_L} C \vee (A \to B)$, then $\vDash_{G3_L} C \vee (\neg B \to \neg A)$, as was to be proved. □

Finally, we have:

Proposition 4.16 (Dcon is an admissible rule of G3$_L$). *The rule Dcon is an admissible rule of G3$_L$. That is, for any A, B, $C \in \mathcal{F}$, (Dcon) $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$.*

Proof. Immediate by Proposition 4.15 and the soundness and completeness theorems (Theorem 4.9 and Theorem 4.13(2)). □

5. Strong soundness and completeness of G3

In this section, it is proved that G3 is strongly sound and complete w.r.t. the RM-semantics defined in the preceding section. Or, equivalently put, we prove that the logic G3^1 is sound and complete w.r.t. the relation $\vDash^1_{G3}$ defined below. The development of G3 along these lines will set the pattern for the treatment of the logics G3^1_L and G3$^\leq_L$ in the following section.

5.1. Types of consequence relations

We shall consider two types of syntactical as well as of semantical consequence relations. But, in order to introduce them, we need a preliminary definition.

Definition 5.1 (Disjunctive rules). *Let S be a propositional logic (cf. Remark 1.1) and r. $A_1, ..., A_n \Rightarrow B$ be a rule of S. The disjunctive rule corresponding to r, Dr, is the following (C is any wff), (Dr) $C \vee A_1, C \vee A_2, ..., C \vee A_n \Rightarrow C \vee B$.*

Now, in some cases, we need to add to a logic S the rule Dr corresponding to the rule r of S if the "thesis form" of r is not a theorem of S. This may in particular be the case of $\mathrm{G3_L}$, where the thesis form of Con, i.e., $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ is not a theorem of $\mathrm{G3_L}$. In Remark 6.20 it is explained why this addition has to be made sometimes and why it is especially necessary when proving completeness in the RM-semantics. Anyway, this eventuality shall be taken into consideration in the proof-theoretical definitions to follow.

Now, let S be a propositional logic (cf. Remark 1.1) and suppose that an RM-semantics has been defined for S. Furthermore, unless otherwise stated, let Γ and A refer to any set of wffs and any wff, respectively, throughout this and the following section. Then, we set:

Definition 5.2 (Proof-theoretical consequence relation. First sense). $\Gamma \vdash^a_S A$ *("A is a-derivable from Γ in S" or "A is derivable from Γ in a first sense") iff there is a finite sequence of wffs $B_1, ..., B_n$ such that B_n is A and for each B_i $(1 \leq i \leq n)$ one of the following is the case: (1) $B_i \in \Gamma$; (2) B_i is an axiom of S; (3) B_i is the result of applying any of the primitive rules of derivation of S to one or more previous formulas in the sequence; (4) B_i is the result of applying any of the disjunctive rules of derivation (corresponding to the primitive rules of derivation), if present, to one or more previous formulas in the sequence.*

Definition 5.3 (Proof-theoretical consequence relation. Second sense). $\Gamma \vdash^b_S A$ *("A is b-derivable from Γ in S" or "A is derivable from Γ in a second sense") iff there is a finite sequence of wffs $B_1, ..., B_n$ such that B_n is A and for each B_i $(1 \leq i \leq n)$ one of the following is the case: (1) $B_i \in \Gamma$; (2) B_i is a theorem of S; (3) B_i is the result of applying the rule Adj to two previous formulas in the sequence; (4) B_i has been derived by S-imp (cf. Definition 2.1 about the rules Adj and S-imp. The rule S-imp reads: for any wffs A, B, $\vdash_S A \rightarrow B \ \& \ A \Rightarrow B$).*

Concerning the semantical relations, we define the first one in the present section and the second one in the following section.

Definition 5.4 (Semantical consequence relation. First sense). $\Gamma \vDash_{KS} A$ *("A is a semantical consequence of Γ in S w.r.t. the set K in S-models" or "A is a semantical consequence of Γ in S in a first sense") iff if $a \vDash \Gamma$, then $a \vDash A$ for all $a \in K$ in all S-models ($a \vDash \Gamma$ iff $a \vDash B$ for all $B \in \Gamma$).*

Thus, in the case of G3, we have:

Definition 5.5 (The proof-theoretical relation $\vdash^1_{\mathbf{G3}}$). *$\Gamma \vdash^1_{G3} A$ ("A is derivable from Γ in G3") iff $\Gamma \vdash^a_{G3} A$. That is, $\Gamma \vdash^1_{G3} A$ iff A is a-derivable from Γ in G3.*

Remark 5.6 (No disjunctive rules in $\vdash^1_{\mathbf{G3}}$). *The introduction of disjunctive rules in order to define the relation $\vdash^1_{G3}$ is out of the question since the theses corresponding to the rules Adj, MP, Pref and Suf (i.e., A1, T1, T2 and T3. See the appendix) are theorems of G3.*

Definition 5.7 (The semantical relation $\vDash^1_{\mathbf{G3}}$). *$\Gamma \vDash^1_{G3} A$ ("A is a semantical consequence of Γ in G3") iff $\Gamma \vDash_{KG3} A$. That is, $\Gamma \vDash^1_{G3} A$ iff A is a semantical consequence of Γ w.r.t. K in G3-models.*

5.2. Soundness and completeness of G3

Next, we prove:

Theorem 5.8 (Strong soundness of G3). *If $\Gamma \vdash^1_{G3} A$ then $\Gamma \vDash^1_{G3} A$.*

Proof. Let Γ be a set of wffs and A a wff such that $\Gamma \vdash^1_{\mathrm{G3}} A$. The proof of $\Gamma \vDash^1_{\mathrm{G3}} A$ is by induction on the length of the derivation of A from Γ. If $A \in \Gamma$ or if it has been derived by Adj, the proof is trivial. And if A is an axiom of G3, then $a \vDash A$ for any $a \in K$ in all G3-models, as shown in Theorem 4.8. Finally, if A has been derived by MP, Suf or Pref, the proof follows from the fact that the corresponding theses, i.e., the modus ponens axiom, $[A \wedge (A \to B)] \to B$ (T1), the suffixing axiom $(A \to B) \to [(B \to C) \to (A \to C)]$ (T3) and the prefixing axiom $(B \to C) \to [(A \to B) \to (A \to C)]$ (T2) are theorems of G3 and so, G3-valid by Theorem 4.8. Therefore, for any $a \in K$ in any G3-model, if $a \vDash A$, then $a \vDash B$ (by Lemma 4.6(a) — Entailment Lemma) A being the antecedent and B being the consequent of T1, T2 and T3 above. □

Next, we proceed into proving completeness. We need the standard concept of "set of consequences of a set of wffs", which is generally defined as follows (S refers to a propositional logic, as above —cf. Remark 1.1).

Definition 5.9 (The set of consequences in S of a set of wffs). *The set $C_n\Gamma[S]$ ("the set of all consequences of Γ in S") is defined as follows: $C_n\Gamma[S] = \{A : \Gamma \vdash_S A\}$. Notice that $\vdash_S$ can be understood either in the first sense (Definition 5.2) or in the second sense (Definition 5.3).*

And, in particular, we need the following:

Proposition 5.10 ($C_n\Gamma[\mathbf{G3}^1]$ is a regular G3-theory). *Let Γ be a set of wffs. The set $C_n\Gamma[G3^1]$ (i.e., the set $\{A : \Gamma \vdash^1_{G3} A\}$) is a regular G3-theory.*

Proof. We have to prove that $C_n\Gamma[\mathrm{G3}^1]$ is closed under Adj, G3-imp and contains all theorems of G3. Now, it is trivial that $C_n\Gamma[\mathrm{G3}^1]$ is closed under Adj and MP; and it is clear, by Definition of G3 (cf. the appendix) and Definition 5.9 that $C_n\Gamma[\mathrm{G3}^1]$ contains all theorems of G3. Finally, $C_n\Gamma[\mathrm{G3}^1]$

is closed under G3-imp, since it contains all theorems and is closed under MP. Consequently, $C_n\Gamma[\mathrm{G3}^1]$ is a regular G3-theory. □

Finally, we prove completeness.

Theorem 5.11 (Strong completeness of G3). *If* $\Gamma \vDash^1_{G3} A$, *then* $\Gamma \vdash^1_{G3} A$.

Proof. Suppose that Γ is a set of wffs and A a wff such that $\Gamma \nvdash^1_{\mathrm{G3}} A$. Then, $A \notin C_n\Gamma[\mathrm{G3}^1]$. Now, given that $C_n\Gamma[\mathrm{G3}^1]$ is a non-empty G3-theory such that $A \notin C_n\Gamma[\mathrm{G3}^1]$, there is a regular, prime, w-consistent theory x such that $C_n\Gamma[\mathrm{G3}^1] \subseteq x$ and $A \notin x$ (Lemma 2.9). As $\Gamma \subseteq C_n\Gamma[\mathrm{G3}^1]$, $\Gamma \subseteq x$. By Definition 4.10, $x \vDash^C \Gamma$ and $x \nvDash^C A$, whence $\Gamma \nvDash^1_{\mathrm{G3}} A$ by Definition 5.7 and Lemma 4.12, as was to be proved. □

6. An RM-semantics for $\mathbf{G3^1_L}$ and $\mathbf{G3^<_L}$

We provide an RM-semantics for the logics $\mathrm{G3^1_L}$ and $\mathrm{G3^<_L}$. We follow the pattern set on for developing $\mathrm{G3}^1$ in the preceding section. First, $\mathrm{G3^<_L}$ will be investigated.

6.1. The logic $\mathbf{G3^<_L}$

We begin by defining the appropriate consequence relations (as in Section 5, and unless otherwise stated, we shall generally refer by Γ and A to any set of wffs and any wff, respectively).

Definition 6.1 (The proof-theoretical relation $\vdash^<_{\mathbf{G3_L}}$). $\Gamma \vdash^<_{G3_L} A$ *iff* $\Gamma \vdash^b_{G3_L} A$. *That is,* $\Gamma \vdash^<_{G3_L} A$ *iff* A *is b-derivable from* Γ *in* $G3_L$. *(Notice that there is no question of disjunctive rules in Definition 6.1 as* $\vdash^<_{G3_L}$ *is a proof-theoretical relation in the second sense.)*

Definition 6.2 (The semantical relation $\vDash^<_{\mathbf{G3_L}}$). $\Gamma \vDash^<_{G3_L} A$ *(*A *is a semantical consequence of* Γ *in* $G3_L$*) iff* $\Gamma \vDash_{KG3_L} A$. *That is,* $\Gamma \vDash^<_{G3_L} A$ *iff* A *is a semantical consequence of* Γ *w.r.t.* K *in* $G3_L$ *-models.*

Next, we prove soundness.

Theorem 6.3 (Soundness of $\mathbf{G3^<_L}$). *If* $\Gamma \vdash^<_{G3_L} A$ *then* $\Gamma \vDash^<_{G3_L} A$.

Proof. The proof is by induction on the length of the derivation of A from Γ. If $A \in \Gamma$ or A has been derived by Adj, the proof is trivial; and if A is a theorem of $\mathrm{G3_L}$, then the proof follows by Theorem 4.9. So, let us consider the case in which A has been derived by $\mathrm{G3_L}$-imp. Now, let $a \in K$ in an arbitrary $\mathrm{G3_L}$-model and suppose $a \vDash \Gamma$. We have to prove $a \vDash A$. By hypothesis of the case, $\Gamma \vdash^<_{\mathrm{G3_L}} B$ and $\vdash_{\mathrm{G3_L}} B \to A$ for some wff B. By Theorem 4.9, $\vDash_{\mathrm{G3_L}} B \to A$, and by hypothesis of induction, $a \vDash B$. Then, by Lemma 4.6(2), $a \vDash A$, as was to be proved. □

Now, before proving completeness, we record the following:

Proposition 6.4 (**$C_n\Gamma[\mathbf{G3_L}]$ is a regular theory).** *The set $C_n\Gamma[G3_L]$ (i.e., the set $\{A : \Gamma \vdash^{<}_{G3_L} A\}$ is a regular theory.*

Proof. It is immediate. □

Theorem 6.5 (Completeness of $\mathbf{G3^{<}_L}$). *If $\Gamma \vDash^{<}_{G3_L} A$, then $\Gamma \vdash^{<}_{G3_L} A$.*

Proof. It is similar to that for $\mathrm{G3}^1$ in Theorem 5.11 and it is left to the reader. □

6.2. The logic $\mathbf{G3^1_L}$. Models, consequence relations, soundness.

In what follows we investigate the logic $\mathrm{G3^1_L}$. The key point is the definition of the consequence relation $\vDash^1_{\mathrm{G3_L}}$ w.r.t. which the logic $\mathrm{G3^1_L}$ is sound and complete. And in order to define this relation, $\mathrm{G3_L}$-models (cf. Definition 4.2) are modified by introducing, as in relevant logics, a designated subset, O, of the set K (cf. Remark 3.3). Firstly, $\mathrm{G3^1_L}$-models and $\mathrm{G3^1_L}$-validity are defined.

Definition 6.6 ($\mathbf{G3^1_L}$-models). *A $G3^1_L$-model is a structure $(K, O, R, ^*, \vDash)$ where K, R, * and $\vDash$ are defined similarly as in $G3_L$-models (cf. Definition 4.2) and O is a non-empty subset of K such that the following postulate holds: (P13) If $a \in O$, then $Rabc \Rightarrow Rac^*b^*$. Notice that P13 is a restriction of P7 to the set O (P7 does not generally hold in $G3_L$-models).*

Definition 6.7 ($\mathbf{G3^1_L}$-validity). *A formula A is $G3^1_L$-valid (in symbols, $\vDash^1_{G3_L} A$) iff $a \vDash A$ for all $a \in O$ in all $G3^1_L$-models.*

Now, we define the proof-theoretical relation $\vdash^1_{\mathrm{G3_L}}$ and then, the semantical relation $\vDash^1_{\mathrm{G3_L}}$ which will be shown to be coextensive with it.

Definition 6.8 (The proof-theoretical relation $\vdash^1_{\mathbf{G3_L}}$). *$\Gamma \vdash^1_{G3_L} A$ iff $\Gamma \vdash^a_{G3_L} A$. That is, $\Gamma \vdash^1_{G3_L} A$ iff A is a-derivable from Γ in $G3_L$. We remark that the rule Dcon (cf. Proposition 4.16) is one of the rules in $\vdash^1_{G3_L}$ together with Adj, MP, Pref, Suf and Con (cf. Definition of $G3_L$ in Appendix I).*

Let S be a propositional logic (cf. Remark 1.1) and S-semantics an RM-semantics with a designated set O in the models. Then, we set:

Definition 6.9 (Semantical consequence relation. Second sense). *$\Gamma \vDash_{OS} A$ ("A is a semantical consequence of Γ in S w.r.t. the set O in S-models" or "A is a semantical consequence of Γ in S in a second sense") iff if $a \vDash \Gamma$, then $a \vDash A$ for all $a \in O$ in all S-models ($a \vDash \Gamma$ iff $a \vDash B$ for all $B \in \Gamma$).*

In the case of $\mathrm{G3^1_L}$, we have:

Definition 6.10 (The semantical relation $\vDash^1_{\mathbf{G3_L}}$). *(1) Let* Γ *be a non-empty set of wffs and* A *a wff. Then,* $\Gamma \vDash^1_{G3_L} A$ *iff* $\Gamma \vDash_{OG3_L} A$*. That is,* $\Gamma \vDash^1_{G3_L} A$ *iff* A *is a semantical consequence of* Γ *in* $G3_L$ *w.r.t the set* O *in* $G3^1_L$*-models. (2) If* Γ *is the empty set of wffs and* A *a wff, then* $\Gamma \vDash^1_{G3_L} A$ *iff* $\Gamma \vDash_{G3_L} A$*. That is,* $\Gamma \vDash^1_{G3_L} A$ *iff* A *is* $G3_L$*-valid (i.e., iff* $a \vDash A$ *for all* $a \in K$ *in all* $G3_L$*-models —cf. Definition 4.5).*

We recall that semantical relations in the first sense are defined in Definition 5.4 and that the relation $\vDash_{K\mathrm{G3_L}}$ was defined in Definition 6.2.

Remark 6.11 (On the condition 2 in Definition 6.10). *Condition 2 in Definition 6.10 is introduced so that* $\vDash^1_{G3_L}$ *and* $\vDash^{<}_{G3_L}$ *validate the same set of theorems. Notice, regarding this question, that although for any wff* A*, if* $\vDash_{KG3_L} A$*, then* $\vDash_{OG3_L} A$ *is immediate, the converse is not, and in fact, has to be postulated and accounted for in the canonical model (cf. Lemma 6.17).*

Next, we prove soundness.

Theorem 6.12 (Soundness of $\mathbf{G3^1_L}$). *If* $\Gamma \vdash^1_{G3_L} A$ *then* $\Gamma \vDash^1_{G3_L} A$*.*

Proof. The proof is by induction on the length of the derivation of A from Γ. (1) Γ is empty. Now, $\Gamma \vdash^1_{\mathrm{G3_L}} A$ iff there is a finite sequence of wffs $B_1, ..., B_n$ such that B_n is A and each B_i $(1 \leq i \leq n)$ is either an axiom or the result of applying any of the rules MP, Adj, Suf, Pref or Con to one or two previous formulas in the sequence (cf. Definition of $\mathrm{G3_L}$ in the appendix). Then, $\Gamma \vDash^1_{\mathrm{G3_L}} A$ follows by Theorem 4.9 and condition 2 in Definition 6.10. (2) Γ is not empty and $\Gamma \vdash^1_{\mathrm{G3_L}} A$. We shall prove $\Gamma \vDash^1_{\mathrm{G3_L}} A$ for any arbitrary $\mathrm{G3^1_L}$-model. If $A \in \Gamma$ or A has been derived by Adj, the proof is trivial; and if A is an axiom, it is immediate by Theorem 4.9.

Then, concerning MP, Suf and Pref, we have, for any $a \in O$, (a) $a \vDash A \to B$ & $a \vDash A \Rightarrow a \vDash B$; (b) $a \vDash B \to C \Rightarrow a \vDash (A \to B) \to (A \to C)$; (c) $a \vDash A \to B \Rightarrow a \vDash (B \to C) \to (A \to C)$ by Theorem 4.9 and Lemma 4.6 given that the corresponding theses (T1, T3 and T2. See the appendix) are theorems of $\mathrm{G3_L}$.

So, it remains to consider the cases when A has been derived by Con or Dcon. (1) A has been derived by Con. Suppose then that A is of the form $\neg C \to \neg B$ and $\Gamma \vdash^1_{\mathrm{G3_L}} B \to C$. By hypothesis of induction $\Gamma \vDash^1_{\mathrm{G3_L}} B \to C$. Suppose now $a \vDash \Gamma$ for some $a \in O$ and, for reductio, $a \nvDash \neg C \to \neg B$. Then, $a \vDash B \to C$, and on the other hand, $b \vDash \neg C$ and $c \nvDash \neg B$ for b, $c \in K$ such that $Rabc$, whence we have $c^* \vDash B$, and by P13 (cf. Definition 6.6), Rac^*b^*. So, $b^* \vDash C$ (Rac^*b^*, $a \vDash B \to C$, $c^* \vDash B$), i.e., $b \nvDash \neg C$, a contradiction. Consequently, $a \vDash \neg C \to \neg B$, as it was required. (2) A has been derived by Dcon. The proof is similar by using again P13. Thus, the proof of Theorem 6.12 ends with the proof of Dcon. □

6.3. Completeness of $\mathbf{G3^1_L}$

Next, we proceed to proving completeness. Firstly, the canonical model is defined.

Definition 6.13 (The canonical $\mathbf{G3^1_L}$-model). *The canonical $G3^1_L$-model is the structure $(K^C, O^C, R^C, {}^{*C}, \vDash^C)$ where $K^C, R^C, {}^{*C}, \vDash^C$ are defined similarly as in Definition 4.11 and O^C is the set of all (non-empty, prime, w-consistent) sc-consistent $G3_L$ -theories.*

Now, as it was proved in Lemma 4.12, the canonical $\mathrm{G3_L}$-model is in fact a $\mathrm{G3_L}$-model. So, in order to prove that the canonical $\mathrm{G3^1_L}$-model is in fact a $\mathrm{G3^1_L}$-model, it suffices to prove the facts recorded in the two lemmas that follow.

Lemma 6.14 (O^C is not empty). *The set O^C is not empty.*

Proof. Let $G3_{\mathrm{L}}$ be the set of its theorems and A a wff that is not a theorem. By Lemma 2.17, there is a prime, regular, sc-consistent $G3_{\mathrm{L}}$*-theory* x such that $A \notin x$. □

Lemma 6.15 (ℙ13 holds canonically). *The postulate P13 holds canonically. That is, if $a \in O^C$, then $R^C abc \Rightarrow R^C ac^* b^*$ for any b, $c \in K^C$.*

Proof. Let $a \in O^C$ and suppose for b, $c \in K^C$ and A, $B \in \mathcal{F}$, $R^C abc$, $A \to B \in a$ and $A \in c^*$. We have to prove $B \in b^*$. As a is sc-consistent, it is closed under Con (Proposition 2.18). So, $\neg B \to \neg A \in a$. Now, suppose for reductio, $B \notin b^*$. Then, $\neg B \in b$, and so, $\neg A \in c$ ($R^C abc$, $\neg B \to \neg A \in a$, $\neg B \in b$), i.e., $A \notin c^*$, a contradiction. Therefore, $B \in b^*$, and thus, P13 holds canonically. □

Given the preceding lemmas we have:

Corollary 6.16 (The canonical $\mathbf{G3^1_L}$-model is a $\mathbf{G3^1_L}$-model). *The canonical $G3^1_L$-model is indeed a $G3^1_L$-model.*

Proof. Immediate by Lemma 4.12(2), Lemma 6.14 and Lemma 6.15. □

On the other hand, the following lemma accounts canonically for condition 2 in the definition of $\vDash^1_{\mathrm{G3_L}}$ (Definition 6.10).

Lemma 6.17 ($\vDash^C_{O\mathbf{G3_L}} A$ iff $\vDash^C_{K\mathbf{G3_L}} A$). *For any wff A, A belongs to every prime, regular, sc-consistent $G3_L$-theory iff A belongs to every prime, regular, w-consistent $G3_L$-theory.*

Proof. (1) Left to right: it follows by Lemma 2.17: If a is a prime, regular, w-consistent theory without A, there is a prime, regular, sc-consistent theory without A. (2) Right to left: it is immediate since each regular sc-consistent theory is w-consistent (cf. Remark 2.8). □

Now, in order to prove completeness we need the following proposition and primeness lemma.

Proposition 6.18 (Ecq is a derivable rule of $\mathbf{G3^1_L}$). *The rule Ecq (cf. Proposition 2.6) is a derivable rule of $G3^1_L$, i.e., for any $A, B \in \mathcal{F}$, (Ecq) $A \wedge \neg A \vdash^1_{G3_L} B$.*

Proof. By A2, (1) $A \wedge \neg A \vdash^1_{\mathrm{G3_L}} A$ and (2) $A \wedge \neg A \vdash^1_{\mathrm{G3_L}} \neg A$. By A11 and 1, (3) $A \wedge \neg A \vdash^1_{\mathrm{G3_L}} \neg B \to A$. By A17, Con and 3, (4) $A \wedge \neg A \vdash^1_{\mathrm{G3_L}} \neg A \to B$, whence by 2, 4 and MP, (5) $A \wedge \neg A \vdash^1_{\mathrm{G3_L}} B$. □

Lemma 6.19 (Primeness). *If $\Gamma \nvdash^1_{G3_L} A$, then there is some prime theory Θ such that $\Gamma \subseteq \Theta$ and $\Theta \nvdash^1_{G3_L} A$.*

Proof. It is similar to that of Lemma 2.10 (cf. [11], Chap. 4, esp. pp.336-340). □

Remark 6.20 (On Lemma 6.19). *A warning, however, is in order (cf. e.g. the clear notes by Brady in [3], §P.3, pp. 7-9). In the aforementioned work [11], it is shown how to prove a primeness result (similar to that in Lemma 6.19) for logics included in B_+ closed under weak rules. A requirement in the proof is that the logics in question as well as the theories built upon them be closed under the "disjunctive form" of these rules. Now, there is no problem with Adj, MP, Pref and Suf since the corresponding theses to these rules (i.e., A1, T1, T2 and T3) are theorems of $G3_L$. Then, regarding Con, it has been essential to show that Dcon is an admissible rule of $G3_L$ (Proposition 4.16) and, then, to allow the use of Dcon in $\vdash^1_{G3_L}$-derivations (cf. Definition 6.8).*

Theorem 6.21 (Completeness of $\mathbf{G3^1_L}$). *If $\Gamma \vDash^1_{G3_L} A$, then $\Gamma \vdash^1_{G3_L} A$.*

Proof. (1) Γ is empty. Then, let A be a wff such that $\vDash_O A$ (cf. Definition 6.9). By Lemma 6.17, $\vDash_{\mathrm{G3_L}} A$ (cf. Definition 4.5, Definition 6.2), whence by Theorem 4.13(2), $\vdash_{\mathrm{G3_L}} A$, i.e., $\vdash^1_{\mathrm{G3_L}} A$, as was to be proved. (2) Γ is not empty. Then, suppose for some wff A, $\Gamma \nvdash^1_{\mathrm{G3_L}} A$. By Lemma 6.19, there is a prime theory Θ such that $\Gamma \subseteq \Theta$ and $\Theta \nvdash^1_{\mathrm{G3_L}} A$. Then, $A \notin \Theta$. So, Θ is regular (Proposition 2.3) and sc-consistent (Proposition 6.18): if Θ contains a contradiction, then A is derivable. Thus, $\Theta \in O^C$. Now, as $\Gamma \subseteq \Theta$, we have in terms of the canonical $\mathrm{G3_L}$-model (cf. Definition 6.13 and Corollary 6.16), $\Theta \vDash^C \Gamma$ and $\Theta \nvDash^C A$. That is, $\Gamma \nvDash^C_{\mathrm{G3_L}} A$, by Definition 6.10, as was to be proved. □

6.4. $\mathbf{G3^<_L}$ is a paraconsistent logic

We end the paper proving that the logic $\mathrm{G3^<_L}$ is paraconsistent.

As it is well-known, the notion of a paraconsistent logic can be rendered as follows (cf. [4] or [8]).

Definition 6.22 (Paraconsistent logics). *Let $\Vdash$ represent a consequence relation (may it be defined either semantically or proof-theoretically). Then, a logic S is paraconsistent if, for any wff, A, B, the rule (Ecq) $A \wedge \neg A \Vdash B$ does not hold in S.*

In other words, a logic S is paraconsistent if theories built upon S are not necessarily trivial when a contradiction arises. Then, we have:

Proposition 6.23 ($\mathbf{G3^1_L}$, $\mathbf{G3^<_L}$ and paraconsistency). *The logic $G3^<_L$ is paraconsistent, but the logic $G3^1_L$ is not.*

Proof. (1) $\mathrm{G3^1_L}$ is not paraconsistent: immediate by Proposition 6.18 ($\mathrm{G3^1_L}$ is closed under Ecq). (2) $\mathrm{G3^<_L}$ is paraconsistent: consider a $\mathrm{G3_L}$-interpretation I such that $I(p_i) = \frac{1}{2}$ and $I(p_m) = 0$ for the ith and mth propositional variables p_i and p_m. Then $I(p_i \wedge \neg p_i) > I(p_m)$. So, $p_i \wedge \neg p_i \nvDash^<_{\mathrm{G3_L}} p_m$ and, consequently, ECQ does not hold in $\mathrm{G3^<_L}$. □

Sometimes, a logic is defined to be paraconsistent if at least one of its sc-inconsistent theories is not trivial. Now, in this sense we have the following:

Proposition 6.24 (sc-inconsistent theories that are w-consistent).
There are regular, prime, w-consistent, complete $G3_L$-theories that are sc-inconsistent (cf. Definition 2.2).

Proof. Let p_i and p_m be the ith and mth propositional variables and consider the set $y = \{B : \vdash_{\mathrm{G3_L}} A \ \& \ \vdash_{\mathrm{G3_L}} [A \wedge (p_i \wedge \neg p_i)] \to B\}$. It is easy to prove that y is a $\mathrm{G3_L}$-theory. Moreover, it is regular (by A2), but y is sc-inconsistent: $p_i \wedge \neg p_i \in y$. Anyway, y is not trivial: $[(p_i \to p_i) \wedge (p_i \wedge \neg p_i)] \to p_m$ is falsified in $\mathrm{MG3_L}$ (cf. Definition 1.3) by any $\mathrm{G3_L}$-interpretation I such that $I(p_i) = \frac{1}{2}$ and $I(p_m) = 0$. So, $\nvdash_{\mathrm{G3_L}} [(p_i \to p_i) \wedge (p_i \wedge \neg p_i)] \to p_m$ by Theorem 4.9 and, consequently, $p_m \notin y$. Now, by Lemma 2.9, there is a regular, prime, w-consistent $\mathrm{G3_L}$-theory x such that $y \subseteq x$ and $p_m \notin x$. As $p_i \wedge \neg p_i \in y$, x is sc-inconsistent. Finally, x is complete, as it is w-consistent (Proposition 2.4 and Proposition 2.16). □

Finally, we note:

Remark 6.25 (Paraconsisteny of G$n^<$ logics). *It is a corollary of Proposition 6.23(2) that all $Gn^<_L$ logics ($G3^<_L$, $G4^<_L$, ...,$Gn^<_L$, ...,$G\infty^<_L$) are paraconsistent.*

Appendix A.

In this appendix we record the logics treated in this paper together with some theorems of some of them that we have used in the preceding sections. Firstly, we define the positive logic $\mathrm{B_+}$ on which all of them are based. As it is known, Routley and Meyer's basic positive logic B_+ is defined as follows (cf. [11] and references therein).

Definition A.1 (The logic B_+). *The logic B_+ is axiomatized with the following axioms and rules*

Axioms

$A1.\ A \to A$

$A2.\ (A \wedge B) \to A \ / \ (A \wedge B) \to B$

$A3.\ [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

$A4.\ A \to (A \vee B) \ / \ B \to (A \vee B)$

$A5.\ [(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

$A6.\ [A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules:

Modus Ponens (MP): $A \to B \ \& \ A \Rightarrow B.$

Adjunction (Adj): $A \ \& \ B \Rightarrow A \wedge B$

Suffixing (Suf): $A \to B \Rightarrow (B \to C) \to (A \to C)$

Prefixing (Pref): $B \to C \Rightarrow (A \to B) \to (A \to C)$

The logic B_+ is the basic positive logic in Routley and Meyer's ternary relational semantics in the sense that no weaker positive logic can be endowed with a semantics of this type (cf. [11]). If negation is added, then the basic logic (in the same sense) is Sylvan and Plumwood's basic logic BM (cf. [12]).

Definition A.2 (The basic logic BM). *The logic BM is axiomatized by adding the following axioms and rule to B_+:*

$A7.\ (\neg A \wedge \neg B) \to \neg(A \vee B)$

$A8.\ \neg(A \wedge B) \to (\neg A \vee \neg B)$

Contraposition (Con). $A \to B \Rightarrow \neg B \to \neg A$

Now, the logic $BM_{dn,v,f}$ is the result of adding to BM the axiom Dn and the rules Veq and Efq, which are defined below. We shall refer to $BM_{dn,v,f}$ by the abbreviation BM'.

Definition A.3 (The logic BM'). *The logic BM' is axiomatized by adding the following axiom and rules to BM:*

A9. Double negation (Dn). $A \to \neg\neg A$

Verum e quodlibet (Veq). $A \Rightarrow B \to A$

E falso quodlibet (Efq). $A \Rightarrow \neg A \to B$

(Verum e quodlibet means "a true proposition follows from any proposition"; E falso quodlibet means "any proposition follows from a false proposition").

Next, the logics $\mathrm{G3}_{(\mathrm{B}_+)}$ and $\mathrm{G3}_{\mathrm{L}(\mathrm{B}_+)}$ are defined. We shall use the abbreviations G3 and $\mathrm{G3_L}$ to refer to $\mathrm{G3}_{(\mathrm{B}_+)}$ and $\mathrm{G3}_{\mathrm{L}(\mathrm{B}_+)}$, respectively.

Definition A.4 (The logic G3). *The logic G3 can be axiomatized by adding the following axioms to B_+:*

$$\begin{aligned}
&A10.\ [A \to (A \to B)] \to (A \to B)\\
&A11.\ A \to (B \to A)\\
&A12.\ (A \to \neg B) \to (B \to \neg A)\\
&A13.\ \neg A \to (A \to B)\\
&A14.\ (A \vee \neg B) \vee (A \to B)
\end{aligned}$$

Definition A.5 (The logic $\mathbf{G3_L}$). *The logic $G3_L$ can be axiomatized by adding the following axioms and rule to B_+: A9, A10, A11, A14, Con, and in addition,*

$$\begin{aligned}
&A15.\ (A \wedge \neg A) \to (B \vee \neg B)\\
&A16.\ \neg A \to [A \vee (A \to B)]\\
&A17.\ \neg\neg A \to A
\end{aligned}$$

Remark A.6 (On the axiomatization of G3 and $\mathbf{G3_L}$). *In [10], $G3_{(B_+)}$ and $G3_{L(B_+)}$ are axiomatized w.r.t. the positive fragment FD_+ of Anderson and Belnap First Degree Entailments Logic FD (see [1]). The logic FD_+ is the result of restricting B_+ as follows, A3 and A4 are restricted to the rule forms (A3′) $A \to B\ \&\ A \to C \Rightarrow A \to (B \wedge C)$ and (A4′) $A \to C\ \&\ B \to C \Rightarrow (A \vee B) \to C$, respectively, and Pref and Suf to the rule Transitivity (Trans) $A \to B\ \&\ B \to C \Rightarrow A \to C$. Therefore, FD_+ is a sublogic of B_+. Consequently, $G3_{(B_+)}$ and $G3_{L(B_+)}$ can be axiomatized w.r.t. B_+, as they are axiomatized w.r.t. FD_+. Here we choose B_+ because, as remarked above, B_+ is the minimal positive logic in the RM-semantics: no weaker positive logic can be interpreted in this semantics.*

Remark A.7 (BM′ is a sublogic of G3 and $\mathbf{G3_L}$). *The logic BM′ is a sublogic of G3 and $G3_L$.*

Proof. The easiest way is to use the matrices MG3 and $\mathrm{MG3_L}$ (in case a tester is needed, the reader can use that in [7]). □

We end the appendix by listing some theorems of G3 and $\mathrm{G3_L}$ that are used throughout the paper (use, as above, MG3 and $\mathrm{MG3_L}$).

(1) Theorems and rules of both G3 and $G3_L$ are:

$$
\begin{aligned}
\text{A7. } & (\neg A \wedge \neg B) \rightarrow \neg(A \vee B) \\
\text{A8. } & \neg(A \wedge B) \rightarrow (\neg A \vee \neg B) \\
\text{A9. } & A \rightarrow \neg\neg A \\
\text{A10. } & [A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B) \\
\text{A11. } & A \rightarrow (B \rightarrow A) \\
\text{A14. } & (A \vee \neg B) \vee (A \rightarrow B) \\
\text{A15. } & (A \wedge \neg A) \rightarrow (B \vee \neg B) \\
\text{A16. } & \neg A \rightarrow [A \vee (A \rightarrow B)] \\
\text{Veq. } & A \Rightarrow B \rightarrow A \\
\text{Efq. } & A \Rightarrow \neg A \rightarrow B \\
\text{Con. } & A \rightarrow B \Rightarrow \neg B \rightarrow \neg A \\
\text{Dcon. } & C \vee (A \rightarrow B) \Rightarrow C \vee (\neg B \rightarrow \neg A) \\
\text{T1. } & [A \wedge (A \rightarrow B)] \rightarrow B \\
\text{T2. } & (B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)] \\
\text{T3. } & (A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)] \\
\text{T4. } & (A \rightarrow B) \rightarrow [\neg B \vee (\neg B \rightarrow \neg A)] \\
\text{T5. } & (A \rightarrow B) \rightarrow [B \vee (\neg B \rightarrow \neg A)]
\end{aligned}
$$

(2) Theorems and rules of G3:

$$
\begin{aligned}
\text{T6. } & A \rightarrow (\neg A \rightarrow B) \\
\text{T7. } & (A \wedge \neg A) \rightarrow B \\
\text{‘E contradictione quodlibet’ (Ecq). } & A \wedge \neg A \Rightarrow B
\end{aligned}
$$

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