The only 3-valued logic which is a natural implication expansion with the variable-sharing property of Kleene's strong logic

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Abstract

Let us refer by MK3 to Kleene's strong 3-valued matrix. An implicative expansion of MK3 is natural if the conditional function defining it verifies *modus ponens*, assigns a designated value to a conditional whenever it assigns the same value to its antecedent and its consequent, and, finally, it coincides with the classical conditional function when restricted to the "classical" values $\mathbf{t}$ and $\mathfrak{f}$. Two are the main results of this paper. (1) It is proven that, from the viewpoint of functional strength, there is only one 3-valued natural implication expansion of MK3 with the variable-sharing property, the logic we dub $\mathrm{L3^{VSP}}$. (2) It is shown that $\mathrm{L3^{VSP}}$ is a significant and strong logic that can be seen from different perspectives, one of them being to consider it an expansion of classical positive propositional logic.

Keywords: Three-valued logics, Kleene's strong logic, natural conditionals, variable-sharing property, relevant logics

MSC Classification: 03B47

1 Introduction

A propositional logic L has the variable-sharing property (VSP) if in all L-theorems of implication form antecedent and consequent share at least a propositional variable. Given that in propositional logic the non-logical content is conveyed by propositional variables, if L is a propositional logic with the VSP, then it is free from "paradoxes of relevance" in the sense that L does not contain theorems of implication form where the semantical content of antecedent and consequent is disjoint. Anderson and Belnap consider the VSP a necessary property a relevant logic has to fulfill (cf. [1, 4]), but some authors go so far as to consider that "the concept of relevant logic is coextensional with that of having the variable-sharing property" (cf. [19, p. 28]).

On the other hand, Kleene's strong 3-valued matrix MK3 (our label) was defined in [16] in the context of the treatment of partial recursive functions. The matrix MK3 can be rendered as shown in Definition 2.3 below, The connectives are conjunction, disjunction and negation. We take 1 and 2 as designated values (cf. Remark 2.4). Then "1" represents "both true and false", while "2" represents "true only" and "0" stands for "false only".

Finally, the notion of a "natural conditional" is here understood as an extension of that introduced in [27] and can be described as shown in Definition 2.5. That is, a conditional function, $f_{\to}$, expanding MK3 is natural if the three following conditions are fulfilled: (1) $f_{\to}$ coincides with the classical conditional function when restricted to the values 0 and 2; (2) $f_{\to}$ satisfies *modus ponens*; (3) $f_{\to}$ assigns a designated value to a conditional whenever the same value is assigned to its antecedent and consequent (cf. [22]. Of course, there are stricter notions than those defined in [22] and [27]; cf., e.g., [2]).

Now, in [22] it is proved that there are exactly 11 3-valued natural $f_{\to}$-functions expanding MK3 defining relevant conditionals in the sense that $A \to B$ (cf. Definition 2.1) is falsified if A and B do no have at least a variable in common.

Well then, two are the main results of this paper. (1) It is proven that, from the standpoint of functional strength, the 11 implicative expansions of MK3 referred to above actually determine only one logic, since they are functionally equivalent to each other. This unique logic (but cf. the concluding remarks to the paper), unique in the sense just remarked (that is, it is the only 3-valued implication expansion with the VSP of MK3, from the functional viewpoint), is dubbed $\mathrm{L3}^{\mathrm{VSP}}$. (2) It is shown that $\mathrm{L3}^{\mathrm{VSP}}$ is not an artificial construct, but a logic with remarkable properties among which we note, without trying to be exhaustive, the following (cf. also §5 and the appendix to the paper): natural conditionals in the sense of [22]; self-extensionality (i.e., "replacement"); considerable syntactical strength; considerable expressive power (for example, the important 3-valued logics Pac and RM3 —cf., e.g., [15]— are definable in $\mathrm{L3}^{\mathrm{VSP}}$), and finally, its being interpretable in the clear and important two-valued Belnap-Dunn semantics.

The introduction is ended by explaining the structure of the paper. But before doing this, let us point out some notes on many-valued logics with the VSP.

It is known that there are infinitely many logics with the VSP (cf. [12]). Furthermore, some many-valued logics with the VSP have been studied in the literature. For example, the logic determined by Belnap's eight element matrix M_0 (cf. [4], axiomatized in [8]); or the logic characterized by Meyer's six element Crystal lattice, CL, also axiomatized in [8]. But it does not seem easy to interpret the meaning of the logical values in these matrices in an intuitively clear way. However, the meaning of the three truth-values in MK3 and its 3-valued expansions is crystal-clear.

The paper is organized as follows. In §2, some preliminary notions together with the 11 implicative expansions of MK3 with the VSP are defined. In §3 it is proved that the aforementioned 11 implicative expansions are functionally equivalent to each other and so determine only one logic, $L3^{VSP}$. It is also proved that the logics Pac ("Paraconsistency") and RM3 ("the strongest logic in the family of relevance logics" —[2, p. 276]) are functionally included in $L3^{VSP}$ (in addition to [2], cf. [1] and [7] about RM3; [15] and references therein, about Pac). In §4, $L3^{VSP}$ is presented as an expansion of classical positive propositional logic and, finally, in §5, the paper is ended with some concluding remarks on the results obtained and some observations on possible future work to be done in the topic. We have added an appendix including some complementary material as well as a proof of some of the properties $L3^{VSP}$ sports that have been referred to throughout this introduction.

2 The class $MI3^{VSP}$ of implicative expansions of MK3

In this section, we define the class $MI3^{VSP}$ of matrices. The label $MI3^{VSP}$ intends to abbreviate "natural implicative expansions of Kleene's strong matrix MK3 with the variable-sharing property (VSP)". We begin by stating some prior concepts.

Definition 2.1 (Some preliminary notions)**.** *The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives: $\rightarrow$ (conditional or implication[1]), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (negation). The biconditional ($\leftrightarrow$) and the set of formulas (wffs) are defined in the customary way. A, B, C, etc. are metalinguistic variables. Then the ensuing concepts are understood in a fairly standard sense: logical matrix M, M-interpretation, M-consequence and M-validity. Also, the following notions: functions definable in a matrix, functional inclusion and functional equivalence (cf., e.g., [22, §2] or [23]).*

Remark 2.2 (Logics)**.** As suggested in the introduction, in this paper, logics are primarily viewed as M-determined structures, i.e., as structures of the type $(\mathcal{L}, \vDash_M)$ where $\mathcal{L}$ is a propositional language and $\vDash_M$ is a (consequence) relation defined in $\mathcal{L}$ according to the logical matrix M as follows: for any set of wffs Γ and wff A, $\Gamma \vDash_M A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I ($I(\Gamma) \in D$ iff $I(A) \in D$ for all $A \in \Gamma$; D is the set of designated values in M). Thus, from this viewpoint, we can safely travel back and forth from matrices to logics, given the aims of this paper.

Nevertheless, logics are sometimes defined as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from premises" being the usual ones. Furthermore, in

[1] We follow Anderson and Belnap's "Grammatical Propaedeutic", Appendix to [1]: "The principal aim of this piece is to convince the reader that it is philosophically respectable to "confuse" implication and entailment with the conditional, and indeed philosophically suspect to harp on the dangers of such "confusion"" ([1, p. 473].

a derived or secondary sense, we can regard an M-determined logic as a, say, Hilbert-type system (or a natural deduction system or a Gentzen-type system) L such that $\Gamma \vdash_{\mathrm{L}} A$ iff $\Gamma \vDash_{\mathrm{M}} A$, where $\vDash_{\mathrm{M}}$ is the consequence relation defined above and $\Gamma \vdash_{\mathrm{L}} A$ means "A is provable from Γ in L".

Definition 2.3 (Kleene's strong 3-valued matrix)**.** *The propositional language consists of the connectives $\wedge, \vee, \sim$. Kleene's strong 3-valued matrix, MK3 (our label), is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V} = \{0,1,2\}$ with $0 < 1 < 2$; (2) $D = \{1,2\}$; (3) $\mathtt{F} = \{f_\wedge, f_\vee, f_\sim\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice join), respectively, and $f_\sim$ is an involution with $f_\sim(2) = 0, f_\sim(0) = 2$ and $f_\sim(1) = 1$. We display the tables for $\wedge$, $\vee$ and $\sim$:*

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\sim$	0
0	2
1	1
2	0

Remark 2.4 (On designated values in MK3)**.** Kleene does not seem to have considered designated values in [16], §64, although he remarks: "The third "truth-value" $\mathfrak{u}$ is thus not on a par with the other two $\mathfrak{t}$ and $\mathfrak{f}$ in our theory. Consideration of its status will show that we are limited to a special kind of truth-value" ([16, p. 333]). We use $2, 0$ and 1 instead of $\mathfrak{t}, \mathfrak{f}$ and $\mathfrak{u}$, respectively, used by Kleene. The former have been chosen in order to use the tester in [13], in case one is needed. Also, to put in connection the results in the present paper with previous work by us. Finally, we note that the set D can be restricted to $\{2\}$.

On the other hand, we set:

Definition 2.5 (Natural conditionals)**.** *Let $\mathcal{V}$ and D be defined as in Definition 2.3. Then, an $f_\rightarrow$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:*

1. *$f_\rightarrow$ coincides with the $f_\rightarrow$-function for the classical conditional when restricted to the subset $\{0,2\}$ of $\mathcal{V}$.*
2. *$f_\rightarrow$ satisfies modus ponens, that is, for any $a, b \in \mathcal{V}$, if $a \rightarrow b \in D$ and $a \in D$, then $b \in D$.*
3. *For any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a = b$.*

Remark 2.6 (Natural conditionals in Tomova's original paper)**.** We note that natural conditionals are defined in [27] exactly as in Definition 2.5 except for condition (3), which reads there as follows: for any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a \leq b$.

Definition 2.7 (The class $\mathrm{MI3}^{\mathrm{VSP}}$)**.** *In [22, Appendix III, Proposition C.9], it is proved that the only natural implicative expansions of MK3 determining logics (in the sense of Remark 2.2) with the VSP are the ones built up with the conditional described by the following truth tables:*

(t1)

$\rightarrow$	0	1	2
0	2	0	2
1	0	1	0
2	0	0	2

(t2)

$\rightarrow$	0	1	2
0	2	0	2
1	0	1	1
2	0	0	2

(t3)

$\rightarrow$	0	1	2
0	2	0	2
1	0	1	2
2	0	0	2

(t4)

→	0	1	2
0	2	0	2
1	0	1	0
2	0	1	2

(t5)

→	0	1	2
0	2	0	2
1	0	1	0
2	0	2	2

(t6)

→	0	1	2
0	2	1	2
1	0	1	0
2	0	0	2

(t7)

t7	0	1	2
0	2	1	2
1	0	1	0
2	0	1	2

(t8)

→	0	1	2
0	2	1	2
1	0	1	0
2	0	2	2

(t9)

→	0	1	2
0	2	2	2
1	0	1	0
2	0	0	2

(t10)

→	0	1	2
0	2	2	2
1	0	1	0
2	0	1	2

(t11)

→	0	1	2
0	2	2	2
1	0	1	0
2	0	2	2

Then the class MI3^{VSP} is the class of all 3-valued natural implicative expansions of MK3 with the VSP. It consists of the matrics M1, M2,..., M11. Each Mi ($1 \leq i \leq 11$) *is the structure* $(\mathcal{V}, D, \mathtt{F})$ *where* $\mathcal{V}$, D *and* $f_\wedge, f_\vee, f_\sim \in \mathtt{F}$ *are as in Definition 2.3, and* $f_\rightarrow$ *is defined according to table ti.*

Finally, we set:

Definition 2.8 (The LMi-logics)**.** *The logic LMi* ($1 \leq i \leq 11$) *is the logic determined by the matrix Mi in the sense explained in Remark 2.2: for any set of wffs* Γ *and wff* A, $\Gamma \vDash_{LMi} A$ *iff* $I(A) \in \{1,2\}$ *whenever* $I(\Gamma) \in \{1,2\}$ *for all Mi-interpretations* I ($I(\Gamma) \in \{1,2\}$ *iff* $I(A) \in \{1,2\}$ *for all* $A \in \Gamma$*).*

Then the term LMi-logic (s) can be used to refer to the logics determined by the elements in MI3$^{\mathrm{VSP}}$ generally.

In the next section, it is proved that, from the viewpoint of functional strength, the LMi-logics are equivalent different versions of the same logic we name L3$^{\mathrm{VSP}}$ (natural 3-valued logic with the VSP).

3 LM1-LM11 are functionally equivalent

In this section, it is proved that the logics LM1 through LM11 introduced in Definition 2.8 are functionally equivalent to each other. Consequently, these 11 expansions of Kleene's strong logic are just different formulations of the same logic we have named L3$^{\mathrm{VSP}}$ with a different choice of the primitive implication connective (nevertheless, cf. the concluding remarks to the paper). It will also be proved that the 3-valued paraconsistent logic Pac and "the strongest logic in the family of relevance logics" (cf. [2, p. 276]), RM3, are functionally included in L3$^{\mathrm{VSP}}$ (cf. [1, 7] and references therein about RM3; [15] and references therein about Pac).

We begin by investigating the functional relations the logics LM1, LM2, LM3, LM6 and LM9 maintain to each other (Lemmas 3.2-3.5), but firstly, we note a remark on the proofs to follow.

Remark 3.1 (Functions and truth-tables. On displaying proofs of definability)**.** Let f_* be a function defined in $\mathcal{V} = \{0,1,2\}$. In this paper, f_* is usually represented by means of a truth-table t_* (or simply $*$), as for instance, it is the case with $\wedge$, $\vee$ and $\sim$ in MK3 (Definition 2.3). In addition, by k_* (or simply $*$) we refer to the connective defined by t_*. Now, let M be MK3 or an expansion of it. The proof that a given unary or binary function f_* is definable in M is easily visualized by using the connectives corresponding to the functions in M needed in the proof in question. In general, proofs provided below are simplified as just indicated (A, B refer to any wffs —cf. Definition 2.1) On the other hand, in order to prove that an LMi-logic is functionally included in another one, it is clear that it suffices to show that the implication table of the former is definable in the latter, given that we treat only implicative expansions of MK3. (We will simply say "equivalent logics" instead of "functionally equivalent logics", "inclusion" instead of "functional inclusion", etc.) Finally, by $\overset{i}{\to}$ $(1 \leq i \leq 11)$, we refer to the conditional or implication (cf. Note 1) given by table ti. (In case a tester is needed, the one in [13] can be used.)

Lemma 3.2 (LM2 and LM6; LM3 and LM9)**.**

1. *LM2 and LM6 are equivalent to each other.*
2. *LM3 and LM9 are equivalent to each other.*

Proof. We set:

(1a) $A \overset{6}{\to} B =_{\text{df}} \sim B \overset{2}{\to} \sim A$

(1b) $A \overset{2}{\to} B =_{\text{df}} \sim B \overset{6}{\to} \sim A$

(2a) $A \overset{9}{\to} B =_{\text{df}} \sim B \overset{3}{\to} \sim A$

(2b) $A \overset{3}{\to} B =_{\text{df}} \sim B \overset{9}{\to} \sim A$

□

Lemma 3.3 (LM1, LM2 and LM3)**.**

1. *LM1 is included in LM2.*
2. *LM1 is included in LM3.*

Proof. Given Lemma 3.2, we set:

(1) $A \overset{1}{\to} B =_{\text{df}} (A \overset{2}{\to} B) \wedge (A \overset{6}{\to} B)$

(2) $A \overset{1}{\to} B =_{\text{df}} (A \overset{3}{\to} B) \wedge (A \overset{9}{\to} B)$

□

Lemma 3.4 (LM1, LM6 and LM9)**.** *Both LM9 and LM6 are included in LM1.*

Proof. We set:

(1) $A \overset{9}{\to} B =_{\text{df}} A \overset{1}{\to} (A \overset{1}{\to} B)$

(2) $A \overset{6}{\to} B =_{\text{df}} (A \overset{9}{\to} B) \wedge [B \vee (A \overset{1}{\to} B)]$

□

The ensuing corollary follows from the lemmas proved above.

Lemma 3.5 (LM1, LM2, LM3, LM6 and LM9)**.** *The logics LM1, LM2, LM3, LM6 and LM9 are equivalent.*

Proof. The proof is immediate by Lemmas 3.2, 3.3 and 3.4. □

Next, it is shown that LM1 is included in LM4 and in LM5 (Lemma 3.6). Then we investigate the relations between LM4, LM7 and LM10 (Lemma 3.7), on the one hand, and those between LM5, LM8 and LM11, on the other hand (Lemma 3.8).

Lemma 3.6 (LM1, LM4 and LM5)**.** *LM1 is included in both LM4 and LM5.*

Proof. We set:
(1) $A \overset{1}{\to} B =_{\text{df}} (A \overset{4}{\to} B) \wedge (\sim B \overset{4}{\to} \sim A)$
(2) $A \overset{1}{\to} B =_{\text{df}} (A \overset{5}{\to} B) \wedge (\sim B \overset{5}{\to} \sim A)$

□

Lemma 3.7 (LM4, LM7 and LM10)**.** *The logics LM4, LM7 and LM10 are equivalent to each other.*

Proof. We set:
(1) $A \overset{7}{\to} B =_{\text{df}} (A \overset{4}{\to} B) \vee (A \overset{6}{\to} B)$
(2) $A \overset{4}{\to} B =_{\text{df}} (A \overset{7}{\to} B) \wedge [A \vee (\sim B \overset{7}{\to} \sim A)]$
(3) $A \overset{10}{\to} B =_{\text{df}} A \overset{4}{\to} (A \overset{4}{\to} B)$
(4) $A \overset{4}{\to} B =_{\text{df}} (A \overset{10}{\to} B) \wedge [A \vee (\sim B \overset{10}{\to} \sim A)]$
(Notice that table t6 can be used in (1) by virtue of Lemmas 3.5 and 3.6.) □

Lemma 3.8 (LM5, LM8 and LM11)**.** *The logics LM5, LM8 and LM11 are equivalent to each other.*

Proof. We set:
(1) $A \overset{11}{\to} B =_{\text{df}} A \overset{5}{\to} (A \overset{5}{\to} B)$
(2) $A \overset{5}{\to} B =_{\text{df}} (A \overset{11}{\to} B) \wedge [A \vee (\sim B \overset{11}{\to} \sim A)]$
(3) $A \overset{8}{\to} B =_{\text{df}} (A \overset{11}{\to} B) \wedge [B \vee (A \overset{5}{\to} B)$
(4) $A \overset{5}{\to} B =_{\text{df}} (A \overset{8}{\to} B) \wedge [A \vee (\sim B \overset{8}{\to} \sim A)]$

□

In what follows, it is shown that LM4 (resp., LM11) is included in LM1 (resp., LM10). Then LM4 and LM5 are proved equivalent (Lemmas 3.9, 3.10 and 3.11).
Lemma 3.9 (LM1, LM4)**.** *LM4 is included in LM1.*

Proof. Consider the connective $k12$ defined by the ensuing table:

t12	0	1	2
0	0	0	0
1	0	1	0
2	0	1	0

Given LM1, table 4 is defined as follows: $A \overset{4}{\to} B =_{\text{df}} (A \overset{1}{\to} B) \vee (A\ k12\ B)$. So, let us defined t12. We use the connectives $k13, k14$ given by the tables t13, t14.

t13	0	1	2
0	2	0	2
1	0	1	1
2	0	1	2

t14	0	1	2
0	0	1	0
1	2	1	0
2	2	1	0

which are defined as follows: $A\ k13\ B =_{\text{df}} (A \overset{1}{\to} B) \vee (A \wedge B)$; $A\ k14\ B =_{\text{df}} \sim[B \vee (A \overset{1}{\to} B)]$. Then t12 is defined as follows: $A\ k12\ B =_{\text{df}} (A\ k13\ B) \wedge (A\ k14\ B)$ □

Lemma 3.10 (LM11 and LM10)**.** *LM11 is included in LM10.*

Proof. We need the connective $k12$ defined in Lemma 3.9 by using LM1 (notice that LM1 is included in LM4 —Lemma 3.6— and that LM4 and LM10 are equivalent by Lemma 3.7). Also, we use the connectives $k15, k16$ and $k17$ given by the ensuing tables:

t15	0	1	2
0	2	0	2
1	0	1	2
2	2	0	2

t16	0	1	2
0	0	0	0
1	2	1	2
2	0	2	0

t17	0	1	2
0	0	0	0
1	0	1	0
2	0	2	0

which are defined as follows: $A\ k15\ B =_{\mathrm{df}} (A\ k12\ B) \overset{4}{\to} (A \vee B)$; $A\ k16\ B =_{\mathrm{df}} {\sim}({\sim}B\ k15\ {\sim}A)$; $A\ k17\ B =_{\mathrm{df}} (A\ k16\ B)\ k16\ (A\ k12\ B)$. Finally, t11 is defined as follows: $A \overset{11}{\to} B =_{\mathrm{df}} (A \overset{10}{\to} B) \vee (A\ k17\ B)$. □

Lemma 3.11 (LM4 and LM5)**.** *LM4 and LM5 are equivalent.*

Proof. Given Lemmas 3.7, 3.8 and 3.10, it suffices to prove that LM4 is included in LM5. We set: $A \overset{4}{\to} B =_{\mathrm{df}} (A \overset{5}{\to} B \wedge ({\sim}A \vee B)$. □

Finally, we have:

Theorem 3.12 (LM1 through LM11)**.** *LM1 through LM11 are equivalent to each other.*

Proof. Consider the groups of logics (a) = {LM1, LM2, LM3, LM6, LM9}; (b) = {LM4, LM7, LM10}; (c) = {LM5, LM8, LM11}. By Lemmas 3.5, 3.7 and 3.8, it is proved that each one of these groups consists of logics equivalent to each other. Then, given the lemmas just quoted, the logics in (b) and (c) are equivalent by Lemma 3.11; and logics in (a) and in (b) (so in (c)) are equivalent by Lemmas 3.6 and 3.9. Consequently, the 11 LM*i*-logics introduced in Definition 2.8 are equivalent to each other. □

It follows from Theorem 3.12 that $\mathrm{L3}^{\mathrm{VSP}}$ can be defined as the logic determined by any of the matrices in Definition 2.7 (but cf. the concluding remarks to the paper).

The section is ended by showing that the logics Pac and RM3 are included in $\mathrm{L3}^{\mathrm{VSP}}$. These logics are determined by the implicative expansions of MK3 given by the following tables

Pac	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

RM3	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

We have:

Proposition 3.13 (Pac, RM3 and $\mathrm{L3}^{\mathrm{VSP}}$)**.** *The logics Pac and RM3 are included in $L3^{VSP}$.*

Proof. Given Theorem 3.12, the ensuing definitions are sufficient:

(1) $A \overset{\mathrm{Pac}}{\to} B =_{\mathrm{df}} (A \overset{10}{\to} B) \vee (A \overset{3}{\to} B)$

(2) $A \overset{\text{RM3}}{\to} B =_{\text{df}} (A \overset{9}{\to} B) \vee (A \overset{3}{\to} B)$

□

4 L3$^{\text{VSP}}$ as an expansion of classical positive propositional logic

In this section, it is shown how to use the fact that Pac (so classical positive propositional logic) is definable in L3$^{\text{VSP}}$ in order to give easy Hilbert-style formulations (H-formulations) of L3$^{\text{VSP}}$. The H-formulations we define present L3$^{\text{VSP}}$ as an expansion of classical positive propositional logic. We use, say, the implication of LM1 for axiomatizing L3$^{\text{VSP}}$. In the appendix to the paper, an alternative axiomatization is sketched by using that of LM10.

In order to give an H-formulation for L1, we rely upon a strategy based upon Belnap-Dunn two-valued semantics introduced by Brady in [7] (cf. also [8, 9, 21]) as illustrated for 3-valued logics in some papers such as [21, 25].

As it is well-known, Belnap-Dunn two-valued semantics (BD-semantics) is characterized by the possibility of assigning T, F, both T and F or neither T not F to the formulas of a given language (cf. [5, 6, 10, 11]); T represents truth, F represents falsity). Concerning 3-valued logics, two variants of BD-semantics, overdetermined BD-semantics (o-semantics) and underdetermined BD-semantics (u-semantics) can be considered. Formulas can be assigned T, F or both T and F in the former; T, F or neither T nor F in the latter (cf. [21, 25]). U-semantics is especially adequate to 3-valued logics determined by matrices with only one designated value; o-semantics, for those determined by matrices where only one value is not designated.

Given an implicative expansion of MK3, M, with 1 and 2 as designated values, the idea for defining an o-semantics, M_{o}, equivalent to the matrix semantics based upon M is simple: a wff A is assigned both T and F in M_{o} iff A is assigned 1 in M. Next, A is assigned T (resp., F) in M_{o} iff it is not assigned 0 (resp., 2) in M. (Notice that, unlike in u-semantics, interpretation of formulas cannot be empty in o-semantics.)

Then below an o-semantics for LM1 is introduced by defining the notion of an LM1-model and the accompanying notions of LM1-consequence and LM1-validity.

Definition 4.1 (LM1-model)**.** *An LM1-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}\}$, and (ii) I is an LM1-interpretation from the set of all wffs to K, this notion being defined according to the following conditions for each propositional variable p and wffs A, B: (1) $I(p) \in K$; (2a) $T \in I({\sim}A)$ iff $F \in I(A)$; (2b) $F \in I({\sim}A)$ iff $T \in I(A)$; (3a) $T \in I(A \wedge B)$ iff $T \in I(A)$ & $T \in I(B)$; (3b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (4b) $F \in I(A \vee B)$ iff $F \in I(A)$ & $F \in I(B)$; (5a) $T \in I(A \to B)$ iff $[T \notin I(A)$ & $T \notin I(B)]$ or $[T \notin I(A)$ & $F \notin I(B)]$ or $[T \in I(A)$ & $F \in (A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $F \notin I(B)]$; (5b) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(B)$.*

Definition 4.2 (LM1-consequence, LM1-validity)**.** *Let M be an LM1-model. For any set of wffs Γ and wff A:*

1. *$\Gamma \vDash_M A$ (A is a consequence of Γ in M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$. ($T \in I(\Gamma)$ iff $\forall A \in \Gamma (T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma (F \in I(A))$.)*

2. $\Gamma \vDash_{LM1} A$ (*A is a consequence of* Γ *in LM1-semantics) iff* $\Gamma \vDash_M A$ *for each LM1-model M.*
3. *In particular,* $\vDash_{LM1} A$ *(A is valid in LM1-semantics) iff* $\vDash_M A$ *for each LM1-model M (i.e., iff* $T \in I(A)$ *for each LM1-model M).*

By $\vDash_{\text{LM1}}$ we shall refer to the relation just defined.

Now, given Definition 2.7 together with the adjoined notions of M1-interpretation and M1-validity (cf. Definition 2.1) and Definitions 4.1 and 4.2. We easily prove:

Proposition 4.3 (Coextensiveness of $\vDash_{\text{M1}}$ and $\vDash_{\text{LM1}}$)**.** *For any set of wffs* Γ *and a wff A,* $\Gamma \vDash_{M1} A$ *iff* $\Gamma \vDash_{LM1} A$*. In particular,* $\vDash_{M1} A$ *iff* $\vDash_{LM1} A$*.*

Proof. See the proof of Proposition 7.4 in [21] where the simple proof procedure is exemplified in the case of the 24 3-valued natural implicative logics introduced by Tomova in [27]. □

Proposition 4.3 simply formalizes the intuitive translation (explained above) of the matrix semantics based upon M1 into Belnap and Dunn's two-valued type LM1-semantics. Nevertheless, it is a useful proposition, since it gives us the possibility of proving soundness of LM1 w.r.t. $\vDash_{\text{M1}}$ while proving completeness w.r.t. $\vDash_{\text{LM1}}$ by using a canonical model construction.

But let us now define the H-system HLM1. We use $\overset{1}{\rightarrow}$, $\wedge$, $\vee$ and $\sim$ as primitive connectives ($\supset$ is interpreted by table Pac —cf. Proposition 3.13)[2]

Definition 4.4 (The system HLM1)**.** *The system HLM1 can be formulated as follows* ($A_1, ..., A_n \Rightarrow B$ *means "if* $A_1, ..., A_n$*, then B"; the superindex "1" above* $\rightarrow$ *is dropped):*

Axioms:

A1. $A \supset (B \supset A)$
A2. $[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$
A3. $[(A \supset B) \supset A] \supset A$
A4. $(A \wedge B) \supset A;\ (A \wedge B) \supset B$
A5. $A \supset [B \supset (A \wedge B)]$
A6. $A \supset (A \vee B);\ B \supset (A \vee B)$
A7. $(A \supset C) \supset [(B \supset C) \supset [(A \vee B) \supset C]$
A8. $(A \rightarrow B) \supset (A \supset B)$
A9. $\sim\sim A \leftrightarrow A$
A10. $(A \rightarrow B) \leftrightarrow (\sim B \rightarrow \sim A)$
A11. $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
A12. $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
A13. $A \vee \sim A$
A14. $(A \vee B) \vee (A \rightarrow B)$
A15. $(A \vee \sim B) \vee (A \rightarrow B)$

[2]We remark that $\supset$ (as interpreted by Pac —cf. Proposition 3.13) could have been used as a primitive connective instead of any of the 11 conditionals given in Definition 2.7 —cf. Proposition 4.15).

A16. $[A \wedge {\sim}A) \wedge (B \wedge {\sim}B)] \to (A \to B)$
A17. $[(A \to B) \wedge (A \wedge {\sim}A)] \supset {\sim}B$
A18. $[{\sim}(A \to B) \wedge (A \wedge B)] \to ({\sim}A \vee {\sim}B)$
A19. $[{\sim}(A \to B) \wedge ({\sim}A \wedge B)] \to (A \vee {\sim}B)$
A20. $(A \wedge {\sim}A) \supset {\sim}(A \to B)$
A21. $(A \wedge {\sim}B) \supset {\sim}(A \to B)$

Rules of inference:

Modus ponens (MP⊃): $A \supset B, A \Rightarrow B$

Definitions:

$$A \supset B =_{df} A \overset{Pac}{\to} B$$
$$A \leftrightarrow B =_{df} (A \to B) \wedge (B \to A)$$
$$A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$$

Remark 4.5 (On the H-formulation of LM1)**.** We note that if $\wedge$ and $\vee$ are defined as in Łukasiewicz's 3 valued logic Ł3 (cf. [17]), the resulting tables are different from the corresponding ones in MK3. On the other hand, we note that the H-formulation of LM1 is not more complex (in fact, it is simpler) than, say, those for strong 3-valued logics such as G3 (cf. [3] and references therein).

Below, we remark some proof-theoretical properties of LM1.

Proposition 4.6 (Some basic theorems and rules of HLM1)**.** *The following theorems and rules are provable in HLM1:*

1. *Modus ponens for* $\to$ *(MP→):* $A \to B, A \Rightarrow B$
2. *Adjunction (Adj):* $A, B \Rightarrow A \wedge B$
3. *Elimination of* $\wedge$ *(E∧):* $A \wedge B \Rightarrow A, B$
4. *Deduction theorem for* $\supset$ *(DT): If* $\Gamma, A \vdash_{HLM1} B$*, then* $\Gamma \vdash_{HLM1} A \supset B$
5. *If* A *is a classical positive propositional tautology, then* $\vdash_{HLM1} A$*.*
6. *(t1)* $(A \to B) \equiv ({\sim}B \to {\sim}A)$*; (t2)* ${\sim}{\sim}A \equiv A$*; (t3)* ${\sim}({\sim}A \to {\sim}B) \leftrightarrow {\sim}(B \to A)$*; (t4)* ${\sim}({\sim}A \to {\sim}B) \equiv {\sim}(B \to A)$*.*

Proof. It is immediate. (1) By A8 and MP⊃. (2) By A5 and MP⊃. (3) By A4 and MP⊃. (4) By A1 and A2 since MP⊃ is the only rule of inference. (5) By A1 through A7 and MP⊃, as these theses and rule axiomatize classical positive propositional logic. (6) t1-t4 are immediate by A8, A9, A10, E∧, Adj and definitions of $\leftrightarrow$ and $\equiv$. □

By using this proposition we can prove some easy theorems of HLM1 which are instrumental in the completeness proof.

Proposition 4.7 (More theorems of HLM1)**.** *The following theorems are provable in HLM1. (t5)* $({\sim}A \vee {\sim}B) \vee (A \to B)$*; (t6)* $[{\sim}(A \to B) \wedge ({\sim}A \wedge {\sim}B)] \supset (A \vee B)$*; (t7)* $(B \wedge {\sim}B) \supset {\sim}(A \to B)$*; (t8)* $[(A \to B) \wedge {\sim}B] \supset {\sim}A$*; (t9)* $[(A \to B) \wedge (B \wedge {\sim}B)] \supset A$*.*

Proof. It is easy by using Proposition 4.6. In particular, we have: t5: by A14 and t1; t6: by A18, A8 and t2; t7: by A20, t2 and t4; t8: by t1; t9: by A17, t1 and t2. □

In what follows, we proceed to the proofs of soundness and completeness .

Theorem 4.8 (Soundness of HLM1)**.** *For any set of wffs* Γ *and a wff* A*, if* $\Gamma \vdash_{HLM1} A$*, then (1)* $\Gamma \vDash_{M1} A$ *and (2)* $\Gamma \vDash_{LM1} A$*.*

Proof. (1) It is immediate: the axioms of HLM1 are M1 valid and MP$\supset$ preserves M1-validity (recall that the material conditional is understood according to the table $\overset{\text{Pac}}{\to}$: cf. Proposition 3.13; in case a tester is needed, the one in [13] can be used). (2) By (1) and Proposition 4.3. □

Concerning completeness, it is proved by a canonical model construction, as suggested above. Let us see how this proof proceeds. We begin by stating a couple of definitions and a remark.

Definition 4.9 (HLM1-theories)**.** *An HLM1-theory is a set of wffs containing all HLM1-theorems and closed under* $MP_{\supset}$*. An HLM1-theory* t *is prime if whenever* $A \vee B \in t$*, then* $A \in t$ *or* $B \in t$*; and* t *is non-trivial if it does not contain all wffs.*

Remark 4.10 (Complete HLM1-theories)**.** An HLM1-theory t is complete if for any wff A, $A \in t$ or $\sim A \in t$. Now, prime HLM1-theories are complete by virtue of A13.

Definition 4.11 (Canonical HLM1-models)**.** *Let* $\mathcal{T}$ *be a non-trivial prime HLM1-theory. A canonical HLM1-model is the structure* $(K, I_{\mathcal{T}})$ *where (i)* K *is defined as in Definition 4.1 and (ii)* $I_{\mathcal{T}}$ *is a function from the set of all wffs to* K *defined as follows: For each wff* A*,* $T \in I_{\mathcal{T}}(A)$ *iff* $A \in \mathcal{T}$ *and* $F \in I_{\mathcal{T}}(A)$ *iff* $\sim A \in \mathcal{T}$*.*

Then, in order to prove completeness, we have to prove the ensuing two facts:

1. An HLM1-theory without a given wff can be extended to a prime HLM1-theory without the same wff.
2. Let $\mathcal{T}$ be a non-trivial prime HLM1-theory. Then $I_{\mathcal{T}}$ (as defined in Definition 4.11) fulfills clauses (2a), (2b), (3a), (3b) (4a), (4b), (5a) and (5b) (it is immediate that $I_{\mathcal{T}}$ fulfills clause (1)). That is, we have to prove that the canonical translations of clauses (1) through (5b) are provable in $\mathcal{T}$.

We proceed to the proofs of facts 1 and 2.

Lemma 4.12 (Primeness)**.** *Let* A *be a wff and* t *an HLM1-theory such that* $A \notin t$*. Then there is a prime HLM1-theory* $\mathcal{T}$ *such that* $t \subseteq \mathcal{T}$ *and* $A \notin \mathcal{T}$*.*

Proof. It is easy by using classical positive propositional logic (cf., e.g., Lemma 5.9 in [24]). In case that $\wedge$ and $\vee$ are defined by $\supset$ and $\sim$ —which is not the case here—, it suffices to use classical implicative propositional logic (cf., e.g., Lemma 3.9 in [18]). □

Lemma 4.13 (Canonical HLM1-models are HLM1-models)**.** *Let* M_c *be a canonical HLM1-model. Then* M_c *is indeed an HLM1-model.*

Proof. Let $\mathcal{T}$ be a non-trivial prime HLM1-theory and $\mathrm{M_c}$ be the canonical HLM1-model built upon it as indicated in Definition 4.11. In order to prove that $\mathrm{M_c}$ is indeed an HLM1-model it suffices to prove that $I_{\mathcal{T}}$ fulfills clauses (2a) through (5b). We have:

- Clause (2a). It is trivial.
- Clause (2b). By using A9.

- Clause (3a). By A4 and A5.
- Clause (3b). By A12.
- Clause (4a). By primeness of $\mathcal{T}$ and A6.
- Clause (4b). By A11.
- Clause (5a). (⇒) Suppose $A \to B \in \mathcal{T}$. We have to show that at least one of the following alternative obtains: $[A \notin \mathcal{T}$ & $B \notin \mathcal{T}]$ or $[A \notin \mathcal{T}$ & $\sim B \notin \mathcal{T}]$ or $[A \in \mathcal{T}$ & $\sim A \in \mathcal{T}$ & $B \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$ or $[\sim A \notin \mathcal{T}$ & $\sim B \notin \mathcal{T}]$. For *reductio*, suppose that there are wffs A, B such that (1) $A \in \mathcal{T}$ or $B \in \mathcal{T}$ and (2) $A \in \mathcal{T}$ or $\sim B \in \mathcal{T}$ and (3) $A \notin \mathcal{T}$ or $\sim A \notin \mathcal{T}$ or $B \notin \mathcal{T}$ or $\sim B \notin \mathcal{T}$ and (4) $\sim A \in \mathcal{T}$ or $\sim B \in \mathcal{T}$. We have 32 possibilities to consider, but each one of them contains at least one of the following alternatives: (a) a contradiction, e.g., $A \in \mathcal{T}$ & $A \notin \mathcal{T}$; (b) $A \in \mathcal{T}$ and $B \notin \mathcal{T}$; (c) $\sim B \in \mathcal{T}$ & $\sim A \notin \mathcal{T}$; (d) $A \in \mathcal{T}$, $\sim A \in \mathcal{T}$ and $\sim B \notin \mathcal{T}$ or (e) $B \in \mathcal{T}$, $\sim B \in \mathcal{T}$ and $A \notin \mathcal{T}$. But (b)-(e) are also impossible: (b): by MP→; (c): by t8; (d): by A17; (e): by t9.

  (⇐) Suppose $[A \notin \mathcal{T}$ & $B \notin \mathcal{T}]$ or $[A \notin \mathcal{T}$ & $\sim B \notin \mathcal{T}]$ or $[A \in \mathcal{T}$ & $\sim A \in \mathcal{T}$ & $B \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$ or $[\sim A \notin \mathcal{T}$ & $\sim B \notin \mathcal{T}]$. We have to prove that $A \to B \in \mathcal{T}$ follows from each one of these four alternatives. Now, this is immediate by using A14, A15, A16 and t5, respectively.
- Clause (5b). (⇒) Suppose $\sim(A \to B) \in \mathcal{T}$. We have to prove that at least one of the following alternatives follows: $[B \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$ or $[A \in \mathcal{T}$ & $\sim A \in \mathcal{T}]$ or $[A \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$. Suppose for reductio that there are wffs A, B such that (1) $B \notin \mathcal{T}$ or $\sim B \notin \mathcal{T}$ and (2) $A \notin \mathcal{T}$ or $\sim A \notin \mathcal{T}$ and (3) $A \notin \mathcal{T}$ or $\sim B \notin \mathcal{T}$. We have 8 possibilities to consider, but each one of them contains at least one of the following alternatives: (a) $A \notin \mathcal{T}$ and $\sim A \notin \mathcal{T}$; (b) $B \notin \mathcal{T}$ and $\sim B \notin \mathcal{T}$; (c) $A \notin \mathcal{T}$ and $B \notin \mathcal{T}$; (d) $A \notin \mathcal{T}$ and $\sim B \notin \mathcal{T}$ or (e) $\sim A \notin \mathcal{T}$ and $\sim B \notin \mathcal{T}$. But these five situations are impossible: (a), (b): $\mathcal{T}$ is complete. Then, (c)-(d) are shown untenable by using the completeness of $\mathcal{T}$ and t6, A19 and A18, respectively.

  (⇐) Suppose $[A \in \mathcal{T}$ & $\sim A \in \mathcal{T}]$ or $[B \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$ or $[A \in \mathcal{T}$ & $\sim B \in \mathcal{T}]$. We have to prove that $\sim(A \to B) \in \mathcal{T}$, follows from each one of these three alternatives, which is immediate by using A20, t7 and A21, respectively.

  □

Once Lemmas 4.12 and 4.13 proved, the proof of completeness is straightforward.

Theorem 4.14 (Completeness of HLM1)**.** *For any set of wffs Γ and wff A, (1) if $\Gamma \vDash_{M1} A$, then $\Gamma \vdash_{HLM1} A$; (2) if $\Gamma \vDash_{LM1} A$, then $\Gamma \vdash_{HLM1} A$.*

Proof. Firstly, case (2) is proved. (2) Suppose $\Gamma \nvdash_{\mathrm{HLM1}} A$, i.e., that A is not included in the set of consequences derivable in HLM1 from Γ (in symbols, $A \notin \mathrm{Cn}\Gamma[\mathrm{HLM1}]$). Then, $\mathrm{Cn}\Gamma[\mathrm{HL1}]$ is extended to a prime HLM1-theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. Next, the canonical HLM1-model $\mathrm{M_c} = (K, I_{\mathcal{T}})$ based upon $\mathcal{T}$ is defined, and we have $\Gamma \nvDash_{\mathrm{M_c}} A$, since $T \in I_{\mathcal{T}}(\Gamma)$ (as $T \in I_{\mathcal{T}}(\mathrm{Cn}\Gamma[\mathrm{HLM1}])$ but $T \notin I_{\mathcal{T}}(A)$), whence $\Gamma \nvDash_{\mathrm{LM1}} A$ (by Definitions 4.1 and 4.2), as was to be proved.

(1) It is immediate by (2) and Proposition 4.3. □

The section is ended by proving that $\overset{1}{\to}$, in fact, $\overset{i}{\to}$ $(1 \leq i \leq 11)$, can be replaced by $\overset{\mathrm{Pac}}{\to}$ as a primitive connective.

Proposition 4.15 ($\overset{\text{Pac}}{\to}$ as a primitive connective)**.** *Let Pac be the result of expanding MK3 with the conditional table Pac defined in Proposition 3.13. Then table 6 (so, t_i $(i \leq 1 \leq 11)$ is definable from Pac.*

Proof. Consider the connectives $k18, k19, k20, k21, k22$ and $k23$ defined in the following tables:

	t18
0	2
1	1
2	2

t19	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

t20	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

t21	0	1	2
0	2	1	0
1	0	1	0
2	0	0	0

t22	0	1	2
0	2	1	0
1	0	1	0
2	0	0	2

t23	0	1	2
0	0	0	2
1	0	1	0
2	0	0	0

These connectives are defined as follows:

- $k18\ A =_{\text{df}} \sim(A \wedge \sim A)$
- $A\ k19\ B =_{\text{df}} (A \overset{\text{Pac}}{\to} B) \wedge (\sim B \overset{\text{Pac}}{\to} \sim A)$
- $A\ k20\ B =_{\text{df}} k18\ (A\ k19\ B)\ k19\ (A \wedge B)$
- $A\ k21\ B =_{\text{df}} \sim(A \vee B) \wedge (A\ k19\ B)$
- $A\ k22\ B =_{\text{df}} (A\ k20\ B) \vee (A\ k21\ B)$
- $A\ k23\ B =_{\text{df}} k18\ (A\ k19\ B)\ k19\ (\sim A \wedge B)$

Then, the table $\overset{6}{\to}$ is defined as follows: $A \overset{6}{\to} B =_{\text{df}} (A\ k22\ B) \vee (A\ k23\ B)$. So the 11 tables in Definition 2.7 are definable from Pac (i.e., the implicative expansion of MK3 with table $\overset{\text{Pac}}{\to}$). □

5 Concluding remarks

The paper is ended with some brief concluding remarks on the results obtained and on possible future work to be done on the topic.

The most known 3-valued logics such as Łukasiewicz's Ł3, Gödel G3, the quasi-relevant logic RM3 or the paraconsistent logic Pac (cf., e.g., [15, 21] and references in these papers) lack the VSP. In fact, this property is not even predicable of the natural implicative logics defined in [27]. To the best of our knowledge, [22] is the first item in the literature presenting some instances of 3-valued logics with the VSP. These logics are obtained by extending the notion of a "natural conditional" defined in [27].

It is trivial to build up binary expansions of MK3 with the VSP. Consider the ensuing general table t$*$ (a_i $(1 \leq i \leq 4) \in \{0, 1, 2\}$)

$*$	0	1	2
0	0	a_1	0
1	a_2	2	a_3
2	0	a_4	0

Suppose now that A and B do not share some propositional variables in $A * B$. Let M be any expansion of MK3 built up by adding any of the $*$-functions described in t$*$ and let I be an M-interpretation such that $I(p) = 2$ (resp. $I(p) = 0$) for each propositional variable p in A (resp., in B). Then $I(A * B) = 0$. But it is obvious that none of the $*$-functions of t$*$ can represent a reasonable notion of a conjunction, disjunction or implication connective. However, the 11 implicative expansions of MK3 introduced in [22] cannot be deemed artificial constructs as, we think, it has been shown throughout the paper.

In this paper it has been proved that, from the viewpoint of functional strength, there is only one 3-valued natural (in the sense of [22]) implication expansion with the VSP of Kleene's strong logic, the logic we have dubbed L3^{VSP}. It is our opinion that it has also been shown that L3^{VSP} is a significant and strong logic that, we hope, can be of some use in contexts where relevance, paraconsistency and 3-valued decidability are needed.

There is a number of ways in which the research reported in the present paper could be pursued. We shall limit ourselves to remark two of them, but before doing this, let us stress the point that the fact that LM1 through LM11 are functionally equivalent logics does not necessarily make 10 of them dismissable in favor of a given chosen one, for the same reasons that Łukasiewicz's Ł3 is not dismissable in favor of one of the wealth of functional logics equivalent to it defined in [22].

The two possible paths for developing the results here obtained we propose are these:

1. Investigate the functional relations L3^{VSP} maintain with some significant 3-valued logics such as those treated in [21].
2. Investigate the class of 3-valued non C-extending implication expansions of MK3 with the VSP. The members in this class will be, of course, *contraclassical* logics in the sense of [14] and it is well-known that there are very interesting contraclassical logics such as, e.g., connexive logics (cf., e.g., [28]). (We note that, as a way of an example, a 3-valued non C-extending implication expansion of MK3 with the VSP is briefly treated in the appendix.)

Acknowledgments. This work is funded by the Spanish Ministry of Science and Innovation (MCIN/AEI/ 10.13039/501100011033) under Grant [PID2020-116502GB-I00].

Declarations

- Funding: The authors declare that this work is funded by the Spanish Ministry of Science and Innovation (MCIN/AEI/ 10.13039/501100011033) (Grant number [PID2020-116502GB-I00]).
- Conflict of interest/Competing interests: The authors declare that there are no conflict of interests/competing interests.
- Ethics approval: Not applicable.
- Consent to participate: Not applicable.
- Consent for publication: Not applicable.
- Availability of data and materials: Not applicable.
- Code availability: Not applicable.

- Authors' contributions (applicable for works with multiple authors): Not applicable.

Appendix A The system HLM10

In this appendix, we include some complementary material to the topics investigated in the preceding sections. In particular, the ensuing points are treated: as an additional illustration of how to provide H-formulations for the LMi-logics, LM10 is given such a type of axiomatization; some proof-theoretical properties of LM1, LM10 and LMi-logics in general are noted; a comparison between LM1 and the important relevant logics B and DW is made; it is proved that all LMi-logics lack the Ackermann Property, and finally, we briefly present an instance of a 3-valued non C-extending implication expansion of MK3 with the VSP.

Definition A.1 (LM10-models)**.** *An LM10-model is a structure* (K, I) *where* K *and* I *are defined like in LM1-models (cf. Definition 4.1), except for clauses (5a) and (5b), which now read as follows.*

(5a) $T \in I(A \to B)$ *iff* $T \notin I(A)$ *or* $[T \in I(B)$ & $F \in I(B)]$ *or* $[F \notin I(A)$ & $T \in I(B)]$.

(5b) $F \in (A \to B)$ *iff* $[T \in I(A)$ & $F \in I(A)]$ *or* $[T \in I(A)$ & $F \in I(B)]$.

The notions of LM10-consequence and LM10-validity are defined similarly as in LM1 (cf. Definition 4.2).

Then HLM10 is introduced in the next definition ($\wedge, \vee, \sim$ and $\overset{10}{\to}$ are the primitive connectives —cf. note 2).

Definition A.2 (The system HLM10)**.** *The system HLM10 can be formulated as follows* ($A_1, ..., A_n \Rightarrow B$ *means "if* $A_1, ..., A_n$, *then* B*"; the superindex "10" above* $\to$ *is dropped):*

Axioms:

A1. $A \supset (B \supset A)$

A2. $[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$

A3. $[(A \supset B) \supset A] \supset A$

A4. $(A \wedge B) \supset A;\ (A \wedge B) \supset B$

A5. $A \supset [B \supset (A \wedge B)]$

A6. $A \supset (A \vee B);\ B \supset (A \vee B)$

A7. $(A \supset C) \supset [(B \supset C) \supset [(A \vee B) \supset C)]$

A8. $(A \to B) \supset (A \supset B)$

A9. $\sim\sim A \leftrightarrow A$

A10. $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$

A11. $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$

A12. $A \vee \sim A$

A13. $(A \to B) \vee A$

A14. $B \supset [\sim A \vee (A \to B)]$

A15. $[(A \to B) \wedge (A \wedge \sim A)] \to \sim B$

A16. $(B \wedge \sim B) \supset (A \to B)$
A17. $(A \wedge \sim A) \supset \sim (A \to B)$
A18. $(A \wedge \sim B) \supset \sim (A \to B)$
A19. $[\sim (A \to B) \wedge \sim A] \to A$
A20. $[\sim (A \to B) \wedge (A \wedge B)] \to (\sim A \vee \sim B)$
A21. $[\sim (A \to B) \wedge (\sim A \wedge B)] \to (A \vee \sim B)$

Rules of inference:

Modus ponens (MP⊃): $A \supset B, A \Rightarrow B$

Definitions: $\supset$, $\equiv$ *and* $\leftrightarrow$ *are defined as in HLM1 (cf. Definition 4.4.).*
Below, some proof-theoretical properties of HLM10 are remarked.

Then we note:
Remark A.3 (Some rules of HLM10)**.** Items (1)-(5) in Proposition 4.6 for HLM1 are also provable (in a similar way) in the case of HLM10.

Now, following the pattern developed for HLM1 in §4, it is not difficult to prove that HLM10 is sound and complete w.r.t. both M10-semantics and LM10-semantics. In what follows, we note some proof-theoretical properties of the LMi-logics.
Remark A.4 (Some proof-theoretical properties of the LMi-logics)**.** Consider now the ensuing theses and rules:

(a) $A \to A$; (b) $(A \wedge B) \leftrightarrow (B \wedge A)$; (c) $(A \vee B) \leftrightarrow (B \vee A)$; (d) $[A \wedge (B \wedge C)] \leftrightarrow [(A \wedge B) \wedge C]$; (e) $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$; (f) $[A \wedge (B \vee C)] \leftrightarrow [(A \wedge B) \vee (A \wedge C)]$; (g) $[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (A \vee C)]$; (h) $[A \vee (A \wedge B)] \leftrightarrow A$; (i) $[A \wedge (A \vee B)] \leftrightarrow A$; (j) $A \leftrightarrow (A \wedge A)$; (k) $A \leftrightarrow (A \vee A)$; (l) $A \leftrightarrow \sim\sim A$; (m) $\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$; (n) $\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$; (o) $A \vee \sim A$; (E∧) $A \wedge B \Rightarrow A, B$; (Adj) $A, B \Rightarrow A \wedge B$; (CI∧) $A \to B, A \to C \Rightarrow A \to (A \wedge B)$; (I∨) $A \Rightarrow A \vee B, B \vee A$; (E∨) $A \to C, B \to C \Rightarrow (A \vee B) \to C$; (Trans→) $A \to B, B \to C \Rightarrow A \to C$; (Con↔) $A \leftrightarrow B \Rightarrow \sim B \leftrightarrow \sim A$; (Fac↔) $A \leftrightarrow B \Rightarrow (A \wedge C) \leftrightarrow (B \wedge C)$; (Fac′ ↔) $A \leftrightarrow B \Rightarrow (C \wedge A) \leftrightarrow (C \wedge B)$; (Sum↔) $A \leftrightarrow B \Rightarrow (A \vee C) \leftrightarrow (B \vee C)$; (Sum′ ↔) $A \leftrightarrow B \Rightarrow (C \vee A) \leftrightarrow (C \vee B)$; (Trans↔) $A \leftrightarrow B, B \leftrightarrow C \Rightarrow A \leftrightarrow C$; (Suf↔) $A \leftrightarrow B \Rightarrow (B \to C) \leftrightarrow (A \to C)$; (Pref↔) $A \leftrightarrow B \Rightarrow (C \to A) \leftrightarrow (C \to B)$.

Notice that (a) is the self-identity axiom; (b) through (k) formalize the commutative, associative, absorption, idempotence of ∧ and ∨ and distribution between the two connectives. Furthermore, (l) contains the two double negation axioms, (m) and (n) are the De Morgan laws and, finally, (o) is the principle of excluded middle. On the other hand, the abbreviations preceding the rules summarize the following labels. E∧: elimination of ∧; Adj: adjunction; CI∧: conditioned introduction of ∧; I∨: introduction of ∨; E∨: elimination of ∨; Trans→: transitivity of →; Fac↔, Fac′ ↔ (resp. Sum↔, Sum′ ↔): (two versions of) factor w.r.t. ↔ (resp., summation w.r.t. ↔); Con↔: contraposition w.r.t. ↔; Trans↔: transitivity w.r.t. ↔; Suf↔: suffixing w.r.t. ↔, and, finally Pref↔: prefixing w.r.t. ↔.

We have:

Proposition A.5 (Theses and rules provable in LM1, LM10)**.** *The theses and rules displayed in Remark A.4 are provable in LM1 and LM10.*

Proof. It is easy. It suffices to show that the theses are M1-valid and M10-valid, while the rules preserve M1-validity and M10-validity (if needed, the tester in [13] can be used). Then we apply the completeness theorem. □

Remark A.6 (Theses and rules provable in the LMi-logics)**.** It is easy to check that all the theses and rules listed in Remark A.4 are provable in each one of the LMi-logics, except for LM4 and LM5, where everything is provable save Trans (LM4 and LM5 are non-transitive logics w.r.t. its characteristic implication).

Now, given some of the properties noted in Remark A.4, the ensuing proposition is provable in the LMi-logics:

Proposition A.7 (Replacement)**.** *For any wffs A, B, $A \leftrightarrow B \Rightarrow C[A] \leftrightarrow C[A/B]$, where $C[A]$ is a wff in which A appears and $C[A/B]$ is the result of substituting A by B in $C[A]$ in one or more place where A occurs.*

Proof. Induction on the length of C[A] by using Trans↔, Fac↔, Fac′ ↔, Sum↔, Sum′ ↔, Pref↔, Suf↔ and Con↔. □

The facts noted in Remark A.6 and Proposition A.7 allow us to compare the LMi-logics with "implicative logics" as understood in the classical Polish logical tradition.

Definition A.8 (Implicative logics)**.** *A logic is implicative if it fulfills the ensuing conditions for any wffs A, B, C (cf. [20, pp. 179-180]; [29, p. 228])*

C1. $A \to A$	Reflexivity
C2. $A \to B, A \Rightarrow B$	Modus ponens
C3. $A \Rightarrow B \to A$	Veq
C4. $A \to B, B \to C \Rightarrow A \to C$	Transitivity
C5. $A \leftrightarrow B \Rightarrow C[A] \leftrightarrow C[A/B]$	Replacement

(Veq abbreviates "Verum e quodlibet" —"A true proposition follows from any proposition".)

Now, it follows from Remark A.6 and Proposition A.7 that, leaving aside LM4 and LM5, all LMi-logics comply with all conditions listed in Definition A8, except, of course, C3, since Veq encapsulates an infinity of paradoxes of relevance (LM4 and LM5 fail C3 and C4).

Remark A.9 (LM1, B and DW)**.** Routley and Meyer's basic logic B can be axiomatized with the ensuing axioms and rules of inference:

(a1) $A \to A$; (a2) $(A \wedge B) \to A$; $(A \wedge B) \to B$; (a3) $A \to (A \vee B)$; $B \to (A \vee B)$; (a4) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$;(a5) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$; (a6) $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$; (a7) $A \to {\sim}{\sim}A$; (a8) ${\sim}{\sim}A \to A$; (r1 –Adj) $A, B \Rightarrow A \wedge B$; (r2 –MP) $A \to B, A \Rightarrow B$; (r3 –Suf) $A \to B \Rightarrow (B \to C) \to (A \to C)$; (r4 –Pref) $B \to C \Rightarrow (A \to B) \to (A \to C)$; (r5 –Con) $A \to B \Rightarrow {\sim}B \to {\sim}A$.

Then, the logic DW is the result of replacing in B the rule Con by the contraposition axiom $(A \to B) \to (\sim B \to \sim A)$ (cf. [26, Chapter 4] about B and DW). Now, let B′ (resp., DW′) be the result of replacing in B (resp., in DW) the axioms a2 and a3 by the respective corresponding rule E∧ and I∨ (cf. Remark A.4). Well then, we note that LM1 is an expansion of DW′.

Remark A.10 (Ackermann Property)**.** A logic L has the Ackerman Property (AP) if A contains at least an implicative connective in all L-theorems of the form $A \to (B \to C)$. As pointed out above, the VSP is, according to Anderson and Belnap, a necessary property of any *relevant* logic, but, in addition to the VSP, a logic L has to comply with the AP in order to be an *entailment logic* (cf. [1]). LM1-LM11 lack the AP: the wff $p \to (p \to p)$ (an instance of the "mingle axiom" $A \to (A \to A)$) is provable in each one of them.

Remark A.11 (A 3-valued non C-extending implication expansion of MK3 with the VSP)**.** Consider the non C-extending implication expansion of MK3 with the $f_{\to}$-function given by the ensuing table

$\to$	0	1	2
0	2	2	0
1	0	1	0
2	0	1	2

Let us name LM12 the logic determined by this expansion. We note that LM12 has the VSP: if A and B do not share propositional variables in $A \to B$, then $I(A \to B) = 0$ for any M12-interpretation I such that $I(p) = 1$ (resp., $I(p) = 0$) for each variable p in A (resp., in B). Finally, Pac (so classical positive propositional logic —cf. Proposition 3.13) is definable as follows: $A \overset{\text{Pac}}{\to} B =_{df} B \vee (A \overset{12}{\to} B)$. Then, keeping to the pattern set up for axiomatizing LM1 and LM10, LM12 can be given the ensuing H-formulation as an expansion of classical positive propositional logic (cf. note 2):

Axioms: A1-A12 of HLM10 plus (A13) $(B \wedge \sim B) \supset (A \to B)$; (A14) $(A \wedge \sim A) \supset \sim(A \to B)$; (A15) $(A \wedge B) \supset [\sim A \vee (A \to B)]$; (A16) $(\sim A \wedge \sim B) \supset [A \vee (A \to B)]$; (A17) $(A \wedge \sim B) \supset [\sim A \vee \sim(A \to B)]$; (A18) $(\sim A \wedge B) \supset [\sim B \vee \sim(A \to B)]$; (A19) $(A \to B) \supset (\sim A \vee B)$; (A20) $(A \to B) \supset (A \vee \sim B)$; (A21) $[(A \to B) \wedge \sim A] \supset \sim B$; (A22) $\sim(A \to B) \supset (A \vee B)$; (A23) $[\sim(A \to B) \wedge \sim B] \supset A$; (A24) $\sim(A \to B) \supset (\sim A \vee \sim B)$.

The only rule of inference is *modus ponens* for ⊃. Notice that A18, A20 and A21 are *contraclassical* theses (cf. [14]).

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