Manuscript. The final publication is available at Springer via http://dx.doi.org/10.1007/s11225-013-9471-x
(Studia Logica, 102 (1), 185-217, 2014)

Curry's Paradox, Generalized Modus Ponens Axiom and Depth Relevance

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Abstract

"Weak relevant model structures" (wr-ms) are defined on "weak relevant matrices" by generalizing Brady's model structure $\mathcal{M}_{\mathrm{CL}}$ built upon Meyer's Crystal matrix CL. It is shown how to falsify in any wr-ms the Generalized Modus Ponens axiom and similar schemes used to derive Curry's Paradox. In the last section of the paper we discuss how to extend this method of falsification to more general schemes that could also be used in deriving Curry's Paradox.

Keywords: Curry's Paradox; Depth Relevance; Generalized Modus Ponens axiom; Generalized Contraction rule; weak relevant model structures; relevant logic.

1 Introduction

Consider the *Comprehension Axiom* in the form

$$\text{CA. } \exists y \forall x(x \in y \leftrightarrow A) \ (y \text{ is not free in } A)$$

In [10] it is shown that in any logic S closed by *Modus Ponens* (MP), Elimination of the biconditional (E$\leftrightarrow$), uniform substitution of propositional variables and the *Contraction Law* W,

$$\text{W. } [A \to (A \to B)] \to (A \to B)$$

CA trivializes S. We can proceed as follows:

1.	$x \in x \leftrightarrow (x \in x \to p)$	By CA
2.	$x \in x \to (x \in x \to p)$	E$\leftrightarrow$, 1
3.	$(x \in x \to p) \to x \in x$	E$\leftrightarrow$, 1
4.	$[x \in x \to (x \in x \to p)] \to (x \in x \to p)$	W
5.	$x \in x \to p$	MP, 2, 4
6.	$x \in x$	MP, 3, 5
7.	p	MP, 5, 6

But p is arbitrary. So, S is trivial.

The "depth relevance condition" (drc) is introduced in [5]. The drc is motivated in the referred paper as a necessary condition, stated in syntactic terms, for some paraconsistent logics rejecting the Contraction Law W used in deriving Curry's Paradox as it has been shown above.

The aim of this paper is to show, leaning upon Brady's work [5], how to block off in a general way some theses and rules akin to the Contraction Law W, among which the Generalized Modus Ponens axiom defined below is to be found.

Firstly, we shall precisely define the drc. We begin by noting the following:

Remark 1.1 (Languages and logics) *We shall consider logics formulated in the Hilbert-style form defined on propositional languages with a set of denumerable (propositional) variables and some (or all) of the connectives* $\rightarrow$ *(conditional),* $\rightsquigarrow$ *(deep relevant conditional),* $\wedge$ *(conjunction),* $\vee$ *(disjunction) and* $\neg$ *(negation), the biconditionals* $\leftrightarrow$ *and* $\leftrightsquigarrow$ *being defined in the customary way. The set of wff is also defined in the usual way;* A*,* B*,* C *(possibly with subscripts* $0, 1, ..., n$*), etc., are metalinguistic variables.*

Now, as it is known, the following is, according to Anderson and Belnap, a necessary property of any relevant logic S (see [1]).

Definition 1.2 (Variable-sharing property —vsp) *If* $A \rightarrow B$ *is a theorem of S, then* A *and* B *share at least a propositional variable.*

Then, in [5] Brady strengthens the vsp by introducing the "depth relevant condition". In order to define it, it is first convenient to define the notion of "depth of a subformula within a formula" (see [5] and [8], §11).

Definition 1.3 (Depth of a subformula within a formula) *Let* A *be a wff and* B *a subformula of* A*. Then, "the depth of* B *in* A*" (in symbols,* $d[B, A]$*) is inductively defined as follows:*

1. *B is A. Then, $d[B, A] = 0$.*
2. *B is $\neg C$. Then, $d[C, A] = n$ if $d[\neg C, A] = n$.*
3. *B is $C \wedge D$ ($C \vee D$). Then, $d[C, A] = d[D, A] = n$ if $d[C \wedge D, A] = n$ ($d[C \vee D, A] = n$).*
4. *B is $C \rightarrow D$ ($C \rightsquigarrow D$). Then, $d[C, A] = d[D, A] = n+1$ if $d[C \rightarrow D, A] = n$ ($d[C \rightsquigarrow D, A] = n$).*

So, the depth of a particular occurrence of B in A is the number of nested '$\rightarrow$''s ('$\rightsquigarrow$''s) between this particular occurrence of B and the whole formula A.

The "depth relevance condition" is then defined as follows:

Definition 1.4 (Depth relevance condition —drc) *Let* S *be a propositional logic with connectives* $\rightarrow$*,* $\wedge$*,* $\vee$*,* $\neg$ *(cf. Remark 1.1).* S *has the depth relevance condition (or* S *is a deep relevant logic) if in all theorems of* S *of the form* $A \rightarrow B$ *there is at least a propositional variable* p *common to* A *and* B *such that* $d[p, A] = d[p, B]$.

Remark 1.5 (Rewriting $\to$) *If a logic S has the drc, we can write $A \rightsquigarrow B$ instead of $A \to B$ for each theorem of S of the form $A \to B$.*

Example 1.6 (Depth. Depth relevance) *Consider the following wff*

$$\text{(1) } (p \to \neg q) \to [(\neg r \wedge s) \to [(t \vee u) \to w]]$$
$$\text{(2) } (p \to q) \to [[p \to (q \to r)] \to (p \to r)]$$
$$\text{(3) } [\underline{p} \to (p \to q)] \to (\underline{p} \to q)$$

We have: (a) the variables p, q, r and s have depth 2 in (1); the variables t, u and w have depth 3 in (1); (b) antecedent and consequent of (3) have the underlined p at the same depth (notice that (3) is an instance of the Contraction Law W); (c) antecedent and consequent of (2) do not share variables at the same depth.

Consider now the following rule and thesis:

$$\textit{Contraction rule } (\text{rW}).\ A \to (A \to B) \Rightarrow A \to B$$
$$\textit{Modus Ponens axiom } (\text{MPa}).\ [A \wedge (A \to B)] \to B$$

It is well-known that either rW or MPa suffice to derive Curry's Paradox in naive set theories built upon weak positive logics (see, e.g., [15] and references therein). Moreover, as it is remarked in [5] (pp. 72-73), Curry's Paradox is still derivable, and under the same conditions, from the following generalizations of rW and MPa (we maintain Brady's labels to refer to them):

$$(^*)n.\ A \overset{n+1}{\to} B \Rightarrow A \overset{n}{\to} B$$
$$(^{**})n.\ [A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B$$

where $A \overset{n}{\to} B$ abbreviates $A \to [A \to (... \to (A \to B)...)]$ with n occurrencies of A.

We shall discuss rW, MPa, (*)n and (**)n below.

In the last two pages of his paper, Brady investigates to what extent Curry's Paradox can be avoided by logics satisfying the drc. In order to do this, he labels a Curry-type paradox "basic" if it is derived by using rW or MPa. Then, he notes that basic Curry-type paradoxes are avoided in logics with the drc containing Routley and Meyer's basic logic B (cf. [15], Chap. 8; see Appx. 1) as it is implied by the two following facts: (a) in instances of MPa such as $[p \wedge (p \to q)] \to q$, antecedent and consequent do not share variables at the same depth; and, on the other hand, (b) MPa is derivable if rW is added to B (actually, MPa and rW are equivalent w.r.t. a much weaker system Σ —see the definition of Σ in Appx. 1, and the proof of the fact just mentioned in Appx. 2). Consequently, any logic with the drc and containing B (actually, Σ) lacks W no matter the fact that, as it was shown in Example 1.6, in any instance of W antecedent and consequent share at least a propositional variable at the same depth.

Let us briefly discuss this elimination of the basic Curry-type paradoxes.

As just pointed out, this elimination is possible w.r.t. logics containing the system Σ. But there are strong logics in which rW holds and that lack MPa or even (**)n. For illustration purposes, we have chosen a simple example, the 3-valued logic ΣM5 axiomatized in Appx. 3. The logic ΣM5 and a wealth of its subsystems are strong logics. But, on the other hand, ΣM5 does not include B (actually, it does not include Σ). And, regarding rW and (**)n, ΣM5 has the former but lacks the latter for any $n \geq 1$. Therefore, it is not, in principle, impossible for there to be a sublogic of ΣM5 with the drc and with rW as a rule. Or, in other words, in principle, it is not impossible that there is a sublogic of ΣM5 with the drc in which basic Curry-type paradoxes are derivable. It then follows from the preceding considerations that a general method is needed for eliminating these basic paradoxes; a method not dependent on some basic logic, no matter how weak it is. Let us examine this question by considering the general schemes (*)n and (**)n.

To begin with, notice that for any n, antecedent and consequent of (**)n share variables at the same depth. For example, for $n = 2$, we have:

$$[p \wedge [\underline{p} \to (\underline{\underline{p}} \to q)]] \to \{[\underline{p} \wedge [\underline{\underline{p}} \to (p \to q)]] \to q\}$$

where shared variables between antecedent and consequent at the same depth are underlined/double-underlined. Regarding this question Brady notes the following ([5], p. 73): "It is an open problem whether or not any of the rules (*)n, for $n \geq 2$, or any of the formula schemes (**)n, for $n \geq 2$, are derivable in a logic containing the system B satisfying the depth relevance condition. Note that, since (**)n is derivable from (*)n, it suffices to reject (**)n in order to reject (*)n." (In Appx. 2, (**)n is derived from (*)n, given B_+, since Brady does not include the proof of this fact in his paper —actually, it is proved that (*)n and (**)n are equivalent w.r.t. B_+).

The aim of this paper is to define a general class of logics with the drc lacking (**)n and akin theses. This general class comprises a wide spectrum of logics including weak ones not containing *First Degree Entailment Logic* FD (see Appx. 1) as well as strong logics not contained in R-Mingle (see Appx. 1) or even in classical propositional logic (see Appx. 3). It is to be remarked that all (propositional) deep relevant logics defined by Brady in his works (cf. [8] or [9] and references therein. See Appx. 1) are among the logics belonging to this class. Let us now briefly explain how we shall proceed.

Firstly, we shall remark a couple of notes about Brady's strategy. The aim of [5] is to define the main logic (or logics) with the drc. Brady's strategy consists in restricting with the drc the class of logics with the vsp verified by Meyer's Crystal matrix CL (see Appx. 3 where CL is displayed). Then, he chooses the logic DR (presumably, an abbreviation for 'Depth Relevance') as the preferred one among those defined from CL as indicated (see Appx. 1 for a definition of DR and other relevant logics. We remark that the logic DJ is Brady's preferred logic in subsequent works. See, e.g., [8] or [9]).

On the other hand, in [13], Brady's strategy is generalized by showing how to define a class of deep relevant logics from each weak relevant matrix. "Weak relevant matrices" are defined in [12] and are characterized as matrices verifying only logics with the vsp. (In the referred paper [13] it is proved that there are deep relevant logics not included in R-Mingle —see Appx. 1 and Appx. 3 on this question).

By using the model structures defined in [13] (which are a generalization of that defined in [5], recalled below in Example 2.13) we shall state a very general sufficient condition for a logic to simultaneously have the drc and lack (**)n and schemes of a similar structure. In this way, this paper aims at providing additional support to that provided in [5] for the dcr as a non "ad hoc" condition for paraconsistent logics founding natural naive set theory.

The structure of the paper is as follows. In §2, the basic notions of a "weak relevant matrix" and a "weak relevant model structure" (wr-ms, for short) are defined. In §3, it is proved that wr-ms only verify logics with the drc. In §4, it is proved that (**)n is falsified in any wr-ms and so, that it is unprovable in any deep relevant logic verified by a wr-ms. Actually, we shall prove that general schemes of a certain structure are falsified by wr-ms, and that (**)n is an instance of one of these general schemes. Finally, in §5, we briefly discuss a plan for further work by remarking a couple of notes on how wr-ms could block off other theses or schemes from which Curry's Paradox could be derived. We have added three appendices. The first one features all the logics mentioned throughout the paper. The second one displays the proof that MPa (respectively, (**)n) is equivalent to rW (respectively, (*)n) given the logic Σ (respectively, B_+). It follows then that any wr-ms verifying Routley and Meyer's basic positive logic B_+ falsifies the Generalized Contraction rule (*)n. In the third one, some logical matrices are recorded. Each one of these matrices verify (cf. §2) some main logic or class of logics in the paper. Neither the results of [5] nor these of [13] are presupposed. We have maintained, as much as possible, Brady's notation and terminology, especially when defining wr-model structures.

2 Preliminary definitions

In this section the notions of a "weak relevant matrix" and "weak relevant model structure" defined in [13] are recalled. But the more basic and well-known notion of a "logical matrix" is first revisited, for definiteness (cf. Remark 1.1).

Definition 2.1 (Logical matrices) *A logical matrix M is a structure* $(K, T, F, f_\to, f_\wedge, f_\vee, f_\neg)$ *where*

1. *K is a set.*
2. *T and F are non-empty subsets of K such that $T \cup F = K$ and $T \cap F = \emptyset$.*
3. *$f_\to, f_\wedge$ and $f_\vee$ are binary functions (distinct from each other) on K and $f_\neg$ is a unary function on K.*

It is said that K is the *set of elements of* M; T is the *set of designated elements*, and F, the set *of non-designated elements*. The functions $f_\to, f_\wedge, f_\vee$

and $f_\neg$ interpret in M the conditional, conjunction, disjunction and negation, respectively. In some cases one or more of these functions may not be defined.

Remark 2.2 (On the set F) *The set F has been remarked in Definition 2.1 only because it eases the definition of "weak relevant matrices" and "weak relevant model structures".*

In addition to Definition 2.1 we set (cf. Remark 1.1):

Definition 2.3 (Verification, Falsification) *Let M be a logical matrix and A a wff.*

1. *M verifies A iff for any assignment, v_m, of elements of K to the propositional variables of A, $v_m(A) \in T$. M falsifies A iff M does not verify A.*
2. *If $A_1, ..., A_n \Rightarrow B$ is a rule of derivation of a logic S, M verifies $A_1, ...A_n \Rightarrow B$ iff for any assignment, v_m, of elements of K to the variables of $A_1, ...A_n$, B, if $v_m(A_1) \in T, ..., v_m(A_n) \in T$, then $v_m(B) \in T$. M falsifies $A_1, ...A_n \Rightarrow B$ iff M does not verify it.*
3. *Let S be a propositional logic. M verifies S iff M verifies all axioms and rules of derivation of S.*

Remark 2.4 (Interpretation of formulas of the form $A \rightsquigarrow B$) *Formulas of the form $A \rightsquigarrow B$ are not interpreted by logical matrices but by model structures defined on weak relevant matrices (see Definition 2.9 below).*

Next, "weak relevant matrices" are defined.

Definition 2.5 (Weak relevant matrices) *Let M be a logical matrix in which $a_F \in F$ and $a_1, ..., a_m, b_1, ..., b_n$ are elements of K. And let us designate by K_1 and K_2 the subsets of K $\{a_1, ..., a_m\}$ and $\{b_1, ..., b_n\}$, respectively. The subsets K_1 and K_2 are disjoint and the members of K_1 as well as those elements in K_2 are possibly (but not necessarily) distinct from each other. Finally, the following conditions are fulfilled:*

1. $\forall x \forall y \in K_1 \ f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_\neg(x) \in K_1$
2. $\forall x \forall y \in K_2 \ f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_\neg(x) \in K_2$
3. $\forall x \in K_1 \forall y \in K_2 \ f_\rightarrow(x, y) = a_F$
4. $\forall x \in K_1 \cup K_2 \ f_\rightarrow(x, a_F) = a_F$

Then, it is said that M is a weak relevant matrix (wr-matrix, for short).

Remark 2.6 (On wr-matrices) *The notion of a wr-matrix is introduced in [12] where "strong relevant matrices" are also defined. On the other hand, the class of wr-matrices in Definition 2.5 is a restriction of that in [12] where condition 4 is not included. Regarding this restriction, notice that members of* $K_1 \cup K_2$ *have to be designated if the "self-identity" axiom* $A \to A$ *is verified. If this is the case, then condition 4 necessarily follows if Modus Ponens is in its turn going to be validated.*

Example 2.7 (Some wr-matrices) *The well-known tables of Belnap in [3] (used again in [1] §22.1.3) form a wr-matrix where* $K_1 = \{-2, +2\}, K_2 = \{-1, +1\}$ *and* $a_F = -3$. *Meyer's matrix CL (see Appx. 3) is a wr-matrix with* $K_1 = \{2\}, K_2 = \{3\}$ *and* $a_F = 0$. *Other wr-matrices used in this paper are displayed in Appx. 3.*

Concerning wr-matrices we prove the following basic proposition (cf. Proposition 3.4 in [12]).

Proposition 2.8 (Logics verified by wr-matrices have the vsp) *Let M be a wr-matrix and S a logic verified by it. Then, S has the vsp.*

Proof. Let $A \to B$ be a wff in which A and B do not share propositional variables, and a_i, b_l, some arbitrary elements of K_1 and K_2, respectively. Define an assignment, v_m, such that for each propositional variable p in A (respectively, in B), $v_m(p) = a_i$ (respectively, $v_m(p) = b_l$). Notice that this is a consistent assignment because A and B do not share propositional variables. By induction on the length of A and B and conditions 1, 2 in Definition 2.5, it follows that $v_m(A) = a_i$ and $v_m(B) = b_l$. Then, by condition 3 in the same definition, $v_m(A \to B) = a_F$. Consequently, if M verifies S, then in each theorem of the form $A \to B$, A and B share at least a propositional variable. That is, S has the vsp.

Next, "wr-model structures" and "valuations on wr-model structures" are defined (we follow Brady in [5]). ■

Definition 2.9 (wr-model structures) *Let M be a wr-matrix. A wr-model structure (wr-ms, for short)* $\mathcal{M}_M$ *is the set* $\{M_o, M_1, ...M_n..., M_\omega\}$ *where* $M_o, M_1, ...M_n..., M_\omega$ *are all identical matrices to the wr-matrix M.*

Definition 2.10 (Valuations and interpretations in wr-ms) *Let* $\mathcal{M}_M$ *be a wr-model structure. By* v_i *it is designated a function from the set of propositional variables to K in* M_i $(0 \leq i \leq \omega)$. *Then, a valuation v on* $\mathcal{M}_M$ *is a set of functions* v_i *for each* $i \in \{0, 1, ..., n, ..., \omega\}$. *Given a valuation v, each* v_i *is extended to an interpretation* I_i *of all wff according to the following conditions:*

for all propositional variables p and wff A, B,

$$\begin{aligned}
&(i).\ I_i(p) = v_i(p)\\
&(ii).\ I_i(\neg A) = \neg I_i(A)\\
&(iii).\ I_i(A \wedge B) = I_i(A) \wedge I_i(B)\\
&(iv).\ I_i(A \vee B) = I_i(A) \vee I_i(B)\\
&(v).\ I_i(A \to B) = I_i(A) \to I_i(B)
\end{aligned}$$

where (i)-(v) are calculated according to the wr-matrix M. In addition, formulas of the form $A \rightsquigarrow B$ *are evaluated as follows* ($a_K \in T$. *Cf. Definition 2.5):*

(via). $i = 0 : I_i(A \rightsquigarrow B) = a_K$

(vib). $0 < i < \omega : I_i(A \rightsquigarrow B) = I_{i-1}(A \to B)$

(vic). $i = \omega : I_\omega(A \rightsquigarrow B) \in T$ *iff* $I_j(A \to B) \in T$ *for all* $j (0 \leq j \leq \omega)$

Then the interpretation I on $\mathcal{M}_M$ *extending v is the set of functions* I_i *for each* $i \in \{0, 1, ..., n, ..., \omega\}$.

Next, validity is defined as follows:

Definition 2.11 (Validity in a wr-ms) *Let* $\mathcal{M}_M$ *be a wr-ms and* $B_1, ...B_n, A$, *wff. A is valid in* $\mathcal{M}_M$ *(in symbols* $\vDash_{\mathcal{M}_M} A$*) iff* $I_\omega(A) \in T$ *for all valuations v. And the rule* $B_1, ..., B_n \Rightarrow A$ *preserves* $\mathcal{M}_M$*-validity iff, if* $I_\omega(B_1) \in T, ..., I_\omega(B_n) \in T$, *then* $I_\omega(A) \in T$, *for all valuations v.*

This definition is extended to cover the case of propositional logics in the following.

Definition 2.12 (Logics verified by a wr-ms) *Let* $\mathcal{M}_M$ *be a wr-ms and S a logic (cf. Remark 1.1).* $\mathcal{M}_M$ *verifies S iff all axioms of S are* $\mathcal{M}_M$*-valid and all the rules of S preserve* $\mathcal{M}_M$*-validity.*

Example 2.13 (The wr-ms $\mathcal{M}_{\mathbf{CL}}$) *The wr-ms* $\mathcal{M}_{CL}$ *defined by Brady in [5] is the set* $\{M_o, M_1, ...M_n\ ..., M_\omega\}$ *where* $M_o, M_1, ...M_n..., M_\omega$ *are all identical to CL (cf. Example 2.7 and Appx. 3), valuations are defined w.r.t. the set K of CL, and (i)-(v) are calculated according to the CL-functions as defined in Appx. 3* ($I_0(A \rightsquigarrow B) = 2$ *for all wff A, B). Then, all axioms of DR are* $\mathcal{M}_{CL}$*-valid and all rules of DR preserve* $\mathcal{M}_{CL}$*-validity (cf. Appx. 1. See [5]). Below, it is proved (Theorem 3.3) that any logic verified by a wr-ms is deep relevant. Therefore, DR is a deep relevant logic. On the other hand, it might be interesting to remark (although we cannot pause here to discuss the matter) that the wr-ms* $\mathcal{M}_{M_0}$ *defined on Belnap's wr-matrix* M_0 *recalled in Example 2.7 determines a class of deep relevant logics different from* $\mathcal{M}_{CL}$*. The reason is that the "strong replacement principles", i.e.*

$$\{(A \to B) \wedge [(B \wedge C) \to D]\} \to [(A \wedge C) \to D]$$
$$\{[A \to (B \vee C)] \wedge (C \to D)\} \to [A \to (B \vee D)]$$

and the "strong distribution principle", i.e.,

$$\{[(A \wedge B) \rightarrow C] \wedge [A \rightarrow (B \vee C)]\} \rightarrow (A \rightarrow C)$$

verified by CL are falsified by M_0. We now have a proof (although it cannot be displayed here) that these axioms can be added to DR without it losing the drc (see [15], p. 345 on these priniciples).

In the next section it is proved that wr-ms only verify deep relevant logics.

3 Wr-model structures and the depth relevance condition

We begin by defining a useful notion, "degree of a formula" (in symbols, $deg(A)$ for a wff A), inductively as follows (cf. [5] or [8], §11.1).

Definition 3.1 (Degree of formulas)

1. *If A is a propositional variable, then $deg(A) = 0$.*
2. *If A is of the form $\neg B$ and $deg(B) = n$, then $deg(\neg B) = n$.*
3. *If A is of the form $B \vee C$ ($B \wedge C$) and $deg(B) = n$ and $deg(C) = m$, then $deg(A) = max\{m, n\}$.*
4. *If A is of the form $B \rightarrow C$ ($B \rightsquigarrow C$) and $deg(B) = n$ and $deg(C) = m$, then $deg(A) = max\{m, n\} + 1$.*

So, the degree of a formula A is the maximum number of nested '$\rightarrow$''s ('$\rightsquigarrow$''s) in A.

Next, we shall prove a lemma which will be useful when proving that wr-ms only verify deep relevant logics (see [5], [13]). We shall abbreviate reference to Definition 2.5 and Definition 2.10 by DF.2.5 and DF.2.10, respectively, in this lemma and the theorems that follow it.

Lemma 3.2 (A and B with no variables at the same depth in $A \rightsquigarrow B$) *Let $A \rightsquigarrow B$ be a wff in which A and B do not share variables at the same depth. And let $deg(A \rightsquigarrow B) = n$. Then, $deg(A) \leq n - 1$, $deg(B) \leq n - 1$ (and either $deg(A) = n - 1$ or $deg(B) = n - 1$). On the other hand, let $\mathcal{M}_M$ be a wr-ms. Then, there is an interpretation I on $\mathcal{M}_M$ such that*

1. *For each subformula C of A at depth d, $I_{n-d-1}(C) \in K_1$.*
2. *For each subformula C of B at depth d, $I_{n-d-1}(C) \in K_2$.*

Proof. Let $a_m \in K$, $a_i \in K_1$ and $b_l \in K_2$ and define the valuation v as follows:

1. $v_{n-d-1}(p) = a_i$ for each variable p in A at depth d.

2. $v_{n-d-1}(p) = b_l$ for each variable p in B at depth d.

3. $v_j(p) = a_m$ if $j \geq n$ or $j = n - d - 1$ but p does not occur at depth d neither in A nor in B.

We remark that for each $i \in \{0, 1, ..., n, ..., \omega\}$, v_i has been defined. Now, extend v to I according to clauses (i)-(vi) in DF.2.10 (notice that v is a consistent assignment since A and B do not share variables at the same depth). We prove case (1) (the proof of case (2) is similar). The proof is by induction on the length of C. If C is a propositional variable, then (1) follows by definition of v and condition (i) in DF.2.10. And if C is a formula of the forms $\neg D$, $D \wedge E$, $D \vee E$ or $D \rightarrow E$ the proof is immediate by conditions (ii)-(v) in DF.2.10 (cf. DF.2.5). So, suppose that C is of the form $D \rightsquigarrow C$. By hypothesis of induction, $I_{n-d-2}(D) \in K_1$ and $I_{n-d-2}(E) \in K_1$. That is, $I_{n-d-2}(D \rightarrow E) \in K_1$ by condition (v) in DF.2.10 (cf. DF.2.5). So, by clause (vib) in DF.2.10, $I_{n-d-1}(D \rightsquigarrow E) \in K_1$, as was to be proved. ■

Leaning on the lemma just proved we can show that wr-ms only verify logics with the drc (see [5], [13]).

Theorem 3.3 (wr-ms and the drc) *Let $\mathcal{M}_M$ be a wr-ms and suppose $\vDash_{\mathcal{M}_M}$ $A \rightsquigarrow B$. Then, A and B share a propositional variable at the same depth.*

Proof. Suppose that $A \rightsquigarrow B$ is a wff in which A and B do not share a propositional variable at the same depth, and let $deg(A \rightsquigarrow B) = n$. As A (respectively, B) is a subformula of itself at depth 0 in A (respectively, in B), we have $I_{n-1}(A) \in K_1$ and $I_{n-1}(B) \in K_2$ by Lemma 3.2. So, $I_{n-1}(A \rightarrow B) = a_F$ by condition (v) in DF.2.10 (cf. DF.2.5), whence by condition (vic) in DF.2.10, $I_\omega(A \rightsquigarrow B) \notin T$. Consequently, if $A \rightsquigarrow B$ is valid in the wr-ms $\mathcal{M}_{\mathrm{M}}$, then A and B share at least a propositional variable at the same depth. ■

An immediate corollary is:

Corollary 3.4 (wr-ms and deep relevant logics) *Let $\mathcal{M}_M$ be a wr-ms and S a logic verified by it. Then, S has the drc.*

Proof. Immediate by Theorem 3.3 and Definition 2.12. ■

4 Unprovability of (**)n in (deep relevant) logics verified by wr-ms

Consider t1-t10 below. Although each one of them can appear in logics with the vsp, these are theses containing instances in which antecedent and consequent do not share a propositional variable at the same depth (but for t6 and t7 all these formulas are theorems of relevant logic R see [1] and Appx. 1. Concerning t6 and t7, see [12]).

$$
\begin{array}{rl}
\text{t1.} & (A \to B) \to [(B \to C) \to (A \to C)] \\
\text{t2.} & (B \to C) \to [(A \to B) \to (A \to C)] \\
\text{t3.} & [(A \to A) \to B] \to B \\
\text{t4.} & [A \wedge (A \to B)] \to B \\
\text{t5.} & A \to [(A \to B) \to B] \\
\text{t6.} & A \to (A \to A) \\
\text{t7.} & [(A \to B) \to A] \to A \\
\text{t8.} & (A \to \neg A) \to \neg A \\
\text{t9.} & [(A \to B) \wedge \neg B] \to \neg A \\
\text{t10.} & [(A \to B) \wedge (A \to \neg B)] \to \neg A
\end{array}
$$

Now, Theorem 3.3 guarantees that t1-t10, and formulas similar to them, are ruled out by wr-ms. Let us consider an example.

Example 4.1 (t3 is falsified in the wr-ms CL) *We shall use the matrix CL (see Appx. 3) and the wr-ms $\mathcal{M}_{CL}$ (see Example 2.13). Now, CL verifies t3 (it verifies relevant logic R), but the following instance of t3, when read with $\rightsquigarrow$, i.e.,*

$$t3'.\ [(p \rightsquigarrow p) \rightsquigarrow q] \rightsquigarrow q$$

is not $\mathcal{M}_{CL}$-valid, as it is shown below.

Firstly, notice the following facts: $deg(t3') = 3$, $d[p, (p \rightsquigarrow p) \rightsquigarrow q] = 2$, $d[q, q] = 0$. Next, define the valuation v on $\mathcal{M}_{CL}$ as follows (cf. the proof of Lemma 3.2):

1. *$v_{3-2-1}(p) = 2$; $v_{3-2-1}(r) = 0$ for any propositional variable r distinct from p.*

2. *$v_{3-1-1}(q) = 2$; $v_{3-1-1}(r) = 0$ for any propositional variable r distinct from q.*

3. *$v_{3-0-1}(q) = 3$; $v_{3-0-1}(r) = 0$ for any propositional variable r distinct from q.*

4. *$v_i(p) = 0$ for any propositional variable p when $i \geq 3$.*

Now, notice that v_0, v_1 and v_2 are defined in 1, 2 and 3, respectively; and that v_i for $i \in \{3, ..., n, ..., \omega\}$ is defined in 4. (In 1-4 any element in K instead of 0 will suffice). Next:

5. *Extend v to an interpretation I according to clauses (i)-(vi) in Definition 2.10.*

Then, we show that t3′ is falsified by I. We have $I_0(p \to p) = 2$ (Matrix CL), $I_1(p \rightsquigarrow p) = 2$ (clause (vib), DF.2.10), $I_1[(p \rightsquigarrow p) \to q] = 2$ (Matrix CL), $I_2[(p \rightsquigarrow p) \rightsquigarrow q] = 2$ (clause (vib), DF.2.10), $I_2[[(p \rightsquigarrow p) \rightsquigarrow q] \to q] = 0$ (Matrix CL) and, finally, $I_\omega(t3') \notin T$ (clause (vic), DF.2.10). Therefore, t3′ is not valid in $\mathcal{M}_{CL}$.

So far, so good. But consider t11-t14 below:

t11. $[A \to (A \to B)] \to (A \to B)$
t12. $[A \to (B \to C)] \to [(A \wedge B) \to C]$
t13. $(A \to B) \to [A \to (A \to B)]$
t14. $(A \to B) \to [B \to (A \to B)]$

Antecedent and consequent of any instance of each one of these theses share at least one propositional variable at the same depth. Take, for example, the following instances (shared variables at the same depth are underlined; t11 is, of course, the Contraction Law W):

t11′. $[\underline{p} \to (p \to q)] \to (\underline{p} \to q)$
t12′. $[\underline{p} \to (q \to r)] \to [(\underline{p} \wedge q) \to r]$
t13′. $(\underline{p} \to q) \to [\underline{p} \to (p \to q)]$
t14′. $(p \to \underline{q}) \to [\underline{q} \to (p \to q)]$

Despite this sharing of variables at the same depth, the problem with t11-t14 is that they break the drc when added to weak positive logics with this property. Actually, as pointed out in §1, t11 (or t12) causes this effect if added to such a weak logic as Σ (see Appx. 2). And, on the other hand, t6 above is derivable in any logic with t13 or t14 and the "self-identity" axiom $A \to A$. But, nevertheless, t11-t14 and similar theses are not falsified by Theorem 3.3 that only falsify wff of the form $A \rightsquigarrow B$ when A and B do not share variables at the same depth. In this sense, as remarked in §1, with the following instance of (**)n,

t15. $[p \wedge [\underline{p} \to (\underline{\underline{p}} \to q)]] \to \{[\underline{p} \wedge [\underline{\underline{p}} \to (p \to q)]] \to q\}$

we face the same problem in the case of (*)n or (**)n.

We shall prove, however, that formulas such as t11-t14 and (**)n (for any $n \geq 1$) are valid in no wr-ms. Actually, we shall prove a more general result. Consider the following scheme henceforward labelled t16

t16. $A_1 \rightsquigarrow [A_2 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)]$

where $A_1, ..., A_n, B$ are wff, and no variable p in B such that $d[p, \text{t16}] = s$ also appears in some A_k $(1 \leq k \leq n)$ at depth s in t16. We shall prove that t16 is valid in no wr-ms $\mathcal{M}_{\mathrm{M}}$. Then, it will be proved that (**)n, for any $n \geq 1$, is an instance of t16.

Now, notice that t11-t15 are instances of t16. Also remark that the first occurrence of q, a sentential variable in B, is at depth 2 in t13$'$ whereas the second one is at depth 3 in t13$'$. Finally, notice that it may be the case that A_1 and $A_2 \rightsquigarrow [A_3 \rightsquigarrow (... \rightsquigarrow (An \rightsquigarrow B)...)]$ share one or more variables at the same depth as it actually happens in each one of t11-t15.

After these preliminaries it is time to formulate and prove the following theorem.

Theorem 4.2 (Non-validity of t16) *Let $\mathcal{M}_M$ be a wr-ms, and let*

$$t16.\ A_1 \rightsquigarrow [A_2 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)]$$

be stated as above. That is, for each variable p in B such that $d[p,\ t16] = s$, either (a) p appears in no A_k $(1 \leq k \leq n)$ or else (b) p appears in one or more (or even all) of $A_1, A_2, ..., A_n$ but never having depth s at t16. Then, t16 is not valid in $\mathcal{M}_M$.

Proof. Let $a_m \in K$, $a_i \in K_1$, $b_l \in K_2$ and $deg(\mathrm{t16}) = r$. Then, $deg(A_1) \leq r-1$, $deg(A_2 \rightsquigarrow [A_3 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)]) \leq r-1$; $deg(A_2) \leq r-2$, $deg(A_3 \rightsquigarrow [A_4 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)]) \leq r-2$. In sum, for each k $(1 \leq k \leq n)$, $deg(A_k) \leq r-k$ and $deg(B) \leq r-n$. Now, we define the following valuation v:

1. $v_{r-d-k}(p) = a_i$ for each variable p at depth d in each A_k $(1 \leq k \leq n)$.
2. $v_{r-d-n}(p) = b_l$ for each variable p at depth d in B.
3. $v_j(p) = a_m$ if $j \geq r$ or $j = r-d-k$ $(1 \leq k \leq n)$ but p appears at depth d neither in A_k $(1 \leq k \leq n)$ nor in B.

We remark that v is consistent since no variable in B at depth s in t16 appears in some A_k $(1 \leq k \leq n)$ with $d[p,\ \mathrm{t16}] = s$. Then, extend v to an interpretation I on $\mathcal{M}_\mathrm{M}$ according to clauses (i)-(vi) in Definition 2.10.

Now, similarly as in the proof of Theorem 3.3, by induction on the length of C it is proved:

1. For each subformula C of A_k $(1 \leq k \leq n)$ at depth d, $I_{r-d-k}(C) \in K_1$.
2. For each subformula C of B at depth d, $I_{r-d-n}(C) \in K_2$.

As each A_k $(1 \leq k \leq n)$ (respectively, B) is a subformula of itself at depth 0 in A_k (respectively, in B), we have:

3. $I_{r-k}(A_k) \in K_1$ $(1 \leq k \leq n)$
4. $I_{r-n}(B) \in K_2$.

By condition 3 in DF.2.5 and clause (v) in DF.2.10, we have $I_{r-n}(A_n \rightarrow B) = a_F$; and by clause (vib) in DF.2.10, $I_{r-(n-1)}(A \rightsquigarrow B) = a_F$. By condition 4 in DF.2.5 and clause (v) in DF.2.10, we have $I_{r-(n-1)}(A_{n-1} \rightarrow (A_n \rightsquigarrow B)) =$

a_F, whence, by clause (vib) in DF.2.10, $I_{r-(n-2)}(A_{n-1} \rightsquigarrow (A_n \rightsquigarrow B)) = a_F$. By repeating the argumentation, $I_{r-(n-n)}(A_{n-(n-1)} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)) = a_F$, whence $I_\omega(A_1 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)) \notin T$ by clause (vic) in DF.2.10. Therefore, any formula of the form t16 (fulfilling the conditions stated in Theorem 4.2) is valid in no wr-ms $\mathcal{M}_{\mathrm{M}}$.

Now, Theorem 4.2 can be reformulated in contraposed form as follows. Consider the wff X,

$$\mathrm{X}.\ A_1 \rightsquigarrow (A_2 \rightsquigarrow (... \rightsquigarrow (A_{n-1} \rightsquigarrow A_n)...)$$

where $A_1, ...A_n$ are wff, and let $\mathcal{M}_{\mathrm{M}}$ be any wr-ms such that $\vDash_{\mathcal{M}_{\mathrm{M}}} \mathrm{X}$. Then, for some propositional variable p in A_n such that $d[p,\ \mathrm{X}] = s$, there is some A_k $(1 \leq k \leq n-1)$ in which p appears at depth s in X. ■

Next, it will be shown that (**)n, for any $n \geq 1$, is a formula of the form t16 fulfilling the conditions stated in Theorem 4.2, and it is therefore valid in no wr-ms $\mathcal{M}_{\mathrm{M}}$.

Corollary 4.3 (Non-validity of (**)n in wr-ms) *Let $\mathcal{M}_M$ be a wr-ms. The scheme*

$$(**)n.\ [A \wedge (A \overset{n}{\rightsquigarrow} B)] \overset{n}{\rightsquigarrow} B$$

is not generally valid in $\mathcal{M}_M$.

Proof. We show that there are particular instances of (**)n which are of the form of t16. It would suffice to show that

$$(**)n.\ [p \wedge (p \overset{n}{\rightsquigarrow} q)] \overset{n}{\rightsquigarrow} q$$

(where p and q are distinct propositional variables) is one of them. But we shall prove a more general result.

Consider the scheme

$$\mathrm{t16}'.\ C_1 \rightsquigarrow [C_2 \rightsquigarrow (... \rightsquigarrow (C_n \rightsquigarrow B)...)]$$

where each C_k $(1 \leq k \leq n)$ is of the form $A \wedge (A \overset{n}{\rightsquigarrow} B)$, A and B do not share propositional variables and $deg(B) = 0$ or $deg(B) = 1$. Now, let us refer by B_k to the formula B appearing in C_k $(1 \leq k \leq n)$ and by B_0 to the last occurrence of B in t16′. Then, for each C_k $(1 \leq k \leq n)$, $d[C_k,\ \mathrm{t16}'] = k$; and for each B_k, $d[B_k, C_k] = n$, whence $d[B_k,\ \mathrm{t16}'] = n + k$. On the other hand, $d[B_0,\ \mathrm{t16}'] = n$. Therefore, no variable p in B_0 with $d[p,\ \mathrm{t16}'] = d$ appears in some C_k $(1 \leq k \leq n)$ with the same depth d in t16′. Consequently, Theorem 4.2 applies, and so, t16′ is valid in no wr-ms $\mathcal{M}_{\mathrm{M}}$. Now, it is clear that t16′ is of the form (**)n. So, the scheme (**)n is valid in no wr-ms $\mathcal{M}_{\mathrm{M}}$. ■

An instance of t16′ is for example (**)n' noted above; another is

$$(**)n''.\ [p \wedge (p \overset{n}{\rightsquigarrow} B)] \overset{n}{\rightsquigarrow} B$$

where p does not appear in B and $deg(B) = 0$ or $deg(B) = 1$ (that is, there is at most only one occurrence of $\rightarrow$ ($\rightsquigarrow$) in B). It is to be noted that this condition (i.e., $deg(B) = 0$ or $deg(B) = 1$) is essential in order to falsify t16′. For suppose $deg(B) = k > 1$. If p is a variable appearing in B, then $d[p, B] = m$ where $1 \leq m \leq k$. Now, p can appear in some C_k with $d[p, \text{t16}'] = m$. Consider, for example, $(^{**})n''$ with $n = 2$ and $B = (q \to r) \to r$. That is,

$$\begin{aligned}(^{**})2''.\ & \{p \wedge [p \rightsquigarrow [p \rightsquigarrow [(q \rightsquigarrow r) \rightsquigarrow \underline{r}]]]\} \rightsquigarrow \\ & \{\{p \wedge [p \rightsquigarrow [p \rightsquigarrow [(q \rightsquigarrow r) \rightsquigarrow r]]]\} \rightsquigarrow [(q \rightsquigarrow \underline{r}) \rightsquigarrow r]\}\end{aligned}$$

with the underlined r at depth 4 in $(^{**})2''$: A_1 and B_0 share this variable r at the same depth. (Nevertheless, notice that $(^{**})2''$ is valid in no wr-ms: as no occurrence of r save for the last one is at depth 3, $(^{**})2''$ can be read as an instance of t16.)

We end this section with following corollary.

Corollary 4.4 (Non-validity of (*)n in wr-ms verifying B$_+$) *Let $\mathcal{M}_M$ be a wr-ms verifying B_+. The rule*

$$(^*)n.\ A \overset{n+1}{\rightsquigarrow} B \Rightarrow A \overset{n}{\rightsquigarrow} B$$

does not preserve $\mathcal{M}_M$-validity.

Proof. As shown in Appx. 2, given the logic B+, $(^{**})n$ is derivable from $(^*)n$. So, Corollary 4.4 follows immediately by Corollary 4.3 and the fact just pointed out. ■

To end this section, let us remark that it follows from the theorem and the corollaries just proved that t16, and so, $(^{**})n$ are unprovable in any deep relevant logic verified by a wr-ms; also, that $(^*)n$ is unprovable in any deep relevant logic verified by a wr-ms verifying B$_+$.

5 Conclusions and brief comments on further work

As we have seen, Theorem 4.2 rules out some schemes that would trivialize the *Comprehension Axiom* CA (see §1) when added to weak positive logics, no matter the fact that antecedent and consequent in the said schemes share variables at the same depth. Examples of these formulas and schemes are t11-t16, $(^*)n$ and $(^{**})n$. And the list can be easily extended, and with unexpected theses. Consider, for example, the following wffs

$$\begin{aligned}&\text{t17. } [(\underline{A} \to \underline{B}) \to A] \to [(\underline{A} \to \underline{B}) \to B] \\ &\text{t18. } [(\underline{B} \to \underline{A}) \to A] \to [(\underline{A} \to \underline{B}) \to B] \\ &\text{t19. } [(\underline{A} \wedge B) \to C] \to [(\underline{A} \to (B \to C)]\end{aligned}$$

where metalinguistic variables that can share propositional variables (when substituted by particular wffs) at the same depth are underlined. Theses t17-t19 are invalidated by Theorem 4.2, but they can be theorems of logics with the vsp (see Appx. 3) although t18 is related to Peirce's Law (t7 above in §4) and, most of all, "Positive Paradox" $(A \to (B \to A))$ is immediate if t19 is added to FD_+ (see Appx. 1; on the other hand, t17 is a theorem of E —see Appx. 1).

There are other well-known formulas that being no less rejectable (from the depth relevance perspective) than, for example, t17-t19, are not however invalidated by Theorem 4.2. Some of them are listed below with shared variables at the same depth underlined as above:

t20. $[p \to (q \to \underline{r})] \to [q \to (p \to \underline{r})]$
t21. $[(p \to q) \to (\underline{p} \to \underline{r})] \to [q \to (\underline{p} \to \underline{r})]$
t22. $[(p \to \underline{q}) \to (p \to \underline{r})] \to [p \to (\underline{q} \to \underline{r})]$
t23. $[(p \to q) \to \underline{r})] \to (q \to \underline{r})$
t24. $[(p \vee q) \wedge (q \to \underline{r})] \to [(p \to q) \to \underline{r}]$

Although Theorem 4.2 is not sufficient to rule out these formulas, we can show that t20-t24 (and so, the corresponding schemes) and wffs of similar structure can be invalidated in a similar way as t16 was invalidated. Consider the following:

Theorem 5.1 (Non-validity of t25) *Let $\mathcal{M}_M$ be a wr-ms, and let*

$$t25.\ A_1 \rightsquigarrow [A_2 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)]$$

be a wff where there is some A_k $(1 \leq k \leq n)$ such that for each variable p in A_k with $d[p,\ t25] = s$, we have

1. *p appears neither in B nor in A_m $[m \in \{1, 2, ...k-1, k+1, ...n\}]$*
2. *p appears in some (or all of) $A_1, A_2, ..., A_{k-1}, A_{k+1}, .., A_n, B$, but never having depth s at t25.*

Then, t25 is not valid in $\mathcal{M}_M$.

Proof. (1) $k = 1$. Then, the proof is by Theorem 3.3. (2) $k > 1$. We prove the case $2 \leq k < n$ (if $k = n$, the proof is similar). This proof is like to that of Theorem 4.2. Let $a_p \in K$, $a_i \in K_1$, $b_l \in K_2$ and $deg(\mathrm{t25}) = r$. Then, define the following valuation v:

1. $v_{r-d-k}(p) = a_i$ for each variable p at depth d in A_k.
2. $v_{r-d-m}(p) = b_l$ for each variable p at depth d in each A_m $[m \in \{1, 2, ...k-1, k+1, ...n\}]$.
3. $v_{r-d-n}(p) = b_l$ for each variable p at depth d in B.
4. $v_j(p) = a_p$ if $j \geq r$ or $j = r-d-q$ $(q \in \{1, 2, ..., n\})$ but p does not appear at depth d neither in B nor in A_q $(q \in \{1, 2, ..., n\})$.

Next extend v to an interpretation I on $\mathcal{M}_{\mathrm{M}}$ according to clauses (i)-(vi) in Definition 2.10.

Now, by induction on the length of C, it is proved (cf. Theorem 3.3 and Theorem 4.2):

1. For each subformula C of A_k at depth d, $I_{r-d-k}(C) = a_i$.
2. For each subformula C of A_m $(m \in \{1, 2, ..., k-1, k+1, ...n\})$ at depth d, $I_{r-d-m}(C) = b_l$.
3. For each subformula C of B at depth d, $I_{r-d-n}(B) = b_l$.

Next, as each A_j $(1 \leq j \leq n)$ (respectively, B) is a subformula of itself at depth 0 in A_j (respectively in B), we have:

4. $I_{r-k}(A_k) \in K_1$.
5. $I_{r-m}(A_m) \in K_2$ $(m \in \{1, 2, ..., k-1, k+1, ..., n)$.
6. $I_{r-n}(B) \in K_2$.

Then, we can proceed as follows. By (5) and (6), we have $I_{r-n}(A_n) \in K_2$ and $I_{r-n}(B) \in K_2$, whence, by condition 2 (DF.2.5), $I_{r-n}(A_n \to B) \in K_2$, and, by condition (vib) (DF.2.10), $I_{r-(n-1)}(A_n \rightsquigarrow B) \in K_2$. In this way, by repeating the argumentation, we get $I_{r-(n-(n-k))}(A_{k+1} \rightsquigarrow (A_{k+2} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...))) \in K_2$. Now, $I_{r-(n-(n-k))}(A_k) \in K_1$. So, by condition 3 (DF.2.5), $I_{r-(n-(n-k))}(A_k \to (A_{k+1} \rightsquigarrow (A_{k+2} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)))) = a_F$, and then, $I_{r-(n-(n-(k-1)))}(A_k \rightsquigarrow (A_{k+1} \rightsquigarrow (A_{k+2} \rightsquigarrow ... \rightsquigarrow (A_n \rightsquigarrow B)...))) = a_F$. On the other hand, $I_{r-(n-(n-(k-1)))}(A_{k-1}) \in K_2$. So, by condition 4 (DF.2.5), $I_{r-(n-(n-(k-1)))}(A_{k-1} \to (A_k \rightsquigarrow (A_{k+1} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)))) = a_F$ and, then, $I_{r-(n-(n-(k-2)))}(A_{k-1} \rightsquigarrow (A_k \rightsquigarrow (A_{k+1} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)))) = a_F$. In this way, we obtain, $I_{r-(n-(n-(k-k)))}(A_{k-(k-1)} \rightsquigarrow (A_k \rightsquigarrow (A_{k+1} \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...)))) = a_F$. That is, $I_r(A_1 \rightsquigarrow (A_2 \rightsquigarrow (... \rightsquigarrow (A_n \rightsquigarrow B)...))) = a_F$, whence by condition (vib) (DF.2.10), $I_\omega(16') \notin T$, as was to be proved.

Theorem 5.2 can be reformulated in contraposed form as follows. Let X be the wff

$$\text{X. } A_1 \rightsquigarrow (A_2 \rightsquigarrow (... \rightsquigarrow (A_{n-1} \rightsquigarrow A_n)...))$$

where $A_1, ..., A_n$ are wff and let $\mathcal{M}_{\mathrm{M}}$ be any wr-ms such that $\vDash_{\mathcal{M}_{\mathrm{M}}}$ X. Then, for some propositional variable p in each A_k $(1 \leq i \leq n)$ such that $d[p, A_k] = s$ there is some A_i $(i \in \{1, 2, ..., k-1, k+1, ..., n\})$ in which p appears at depth s in X. ■

Now, notice that t20-t24 are falsified by Theorem 5.1 (one or more variables not appearing at the same depth in the rest of the formula are underlined in each one of t20′-t24′):

$$\begin{aligned}
&\text{t20}'.\ [p \rightsquigarrow (q \rightsquigarrow r)] \rightsquigarrow [\underline{q} \rightsquigarrow (p \rightsquigarrow r)]\\
&\text{t21}'.\ [(p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r)] \rightsquigarrow [\underline{q} \rightsquigarrow (p \rightsquigarrow r)]\\
&\text{t22}'.\ [(p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r)] \rightsquigarrow [\underline{p} \rightsquigarrow (q \rightsquigarrow r)]\\
&\text{t23}'.\ [(p \rightsquigarrow q) \rightsquigarrow r)] \rightsquigarrow (\underline{q} \rightsquigarrow r)\\
&\text{t24}'.\ [(p \vee q) \wedge (q \rightsquigarrow r)] \rightsquigarrow [(\underline{p} \rightsquigarrow \underline{q}) \rightsquigarrow r]
\end{aligned}$$

But t16 and t25 do not exhaust, of course, the class of schemes whose elements would trivialize naive set theory built upon weak positive logics, far from it. In [14], it is recorded "the class of implication formulas *known* to trivialize (NC)" ([14], p. 435; 'NC' abbreviates 'naive comprehension'). Let us examine some of these general structures. Consider the set of schemes (see [14], §5)

$$X_\Psi = \{\prod_{i=1}^{n} A_i \rightarrow q : n \geq 0 \text{ and } A_i \in \vee_\Psi \text{ or a tautology}\}$$

where $\prod_{i=1}^{n} A_i \rightarrow q$ is an abbreviation for $A_1 \rightarrow [A_2 \rightarrow (... \rightarrow (A_n \rightarrow q)...)]$, $\vee_\Psi = \{p \rightarrow \Psi(p,q), \Psi(p,q) \rightarrow p\}$ and $\Psi(p,q)$ is an arbitrary but fixed formula containing no other propositional variables than p and q. In [14] (Theorem 3), it is shown that any wff in X_Ψ trivializes the naive set theory built upon any logic closed under *modus ponens* and uniform substitution of propositional variables. But, unfortunately, wff belonging to X_Ψ are not in general falsifiable by leaning on Theorem 4.2 or on Theorem 5.1. Consider, for example, the case in which $n = 3$ and A_i is of the form $p \rightarrow \Psi(p,q)$, $p \rightsquigarrow (p \rightsquigarrow q)$ in particular, that is

$$\begin{aligned}
X_{\Psi 3}.\ &[p \rightsquigarrow (p \rightsquigarrow q)] \rightsquigarrow\\
&\{[p \rightsquigarrow (p \rightsquigarrow q)] \rightsquigarrow [[p \rightsquigarrow (p \rightsquigarrow q)] \rightsquigarrow q]\}
\end{aligned}$$

Let A be a subformula of $X_{\Psi 3}$. By $\underset{n}{A}$ it is indicated that $\underset{n}{A}$ is the nth occurrence (from left to right) of A in $X_{\Psi 3}$. Then, $X_{\Psi 3}$ is not falsifiable by Theorem 4.2 because

$$d[\underset{4}{q}, X_{\Psi 3}] = d[\underset{1}{q}, X_{\Psi 3}] = 3$$

And $X_{\Psi 3}$ is not falsifiable by Theorem 5.1 because $p \underset{1}{\rightsquigarrow} (p \rightsquigarrow q)$ and $p \underset{2}{\rightsquigarrow} (p \rightsquigarrow q)$ on the one hand, and $p \underset{2}{\rightsquigarrow} (p \rightsquigarrow q)$ and $p \underset{3}{\rightsquigarrow} (p \rightsquigarrow q)$ on the other, share variables at the same depth.

Nevertheless, we remark that $X_{\Psi 3}$ is falsified in Brady's wr-ms $\mathrm{M_{CL}}$ (see Example 2.13). Given that $\underset{1}{p}$ is the only variable at depth 2 in $X_{\Psi 3}$, assign this first occurrence of p the value 0 and all the remaining variables in $X_{\Psi 3}$, no matter the depth of each one of them, the value 3. Then, it is easy to show (cf. Example 4.1) that $X_{\Psi 3}$ has the value 0 for this assignment. Now, this result can be generalized to wr-ms of a certain structure. And so (we think) is the case with the rest of the general classes considered in [14], save for a few exceptions. Therefore, we intend to define general schemes for falsifying the elements in these general classes in adequate wr-ms.

Appendix 1. The logics

The following axioms and rules are formulated in the propositional language described in Remark 1.1. The positive *First Degree Entailment Logic* FD_+ can be formulated as follows (see [1], §15.2).

Axioms:

A1. $A \rightsquigarrow A$

A2. $(A \wedge B) \rightsquigarrow A \ / \ (A \wedge B) \rightsquigarrow B$

A3. $A \rightsquigarrow (A \vee B) \ / \ B \rightsquigarrow (A \vee B)$

A4. $[A \wedge (B \vee C)] \rightsquigarrow [(A \wedge B) \vee (A \wedge C)]$

Rules:

Transitivity (Trans). $A \rightsquigarrow B \ \& \ B \rightsquigarrow C \Rightarrow A \rightsquigarrow C$

Conditioned intro. of conj. (CI $\wedge$). $A \rightsquigarrow B \ \& \ A \rightsquigarrow C \Rightarrow A \rightsquigarrow (B \wedge C)$

Elimination of disjunction (E $\vee$). $A \rightsquigarrow C \ \& \ B \rightsquigarrow C \Rightarrow (A \vee B) \rightsquigarrow C$

The logic Σ is the result of adding the following rules to FD_+:

Adjunction (Adj). $A \ \& \ B \Rightarrow A \wedge B$

Modus ponens (MP). $A \ \& \ A \rightsquigarrow B \Rightarrow B$

Suffixing (Suf). $A \rightsquigarrow B \Rightarrow (B \rightsquigarrow C) \rightsquigarrow (A \rightsquigarrow C)$

The logic *First Degree Entailment* FD is formulated (cf. [1], §15.2) by adding the following axioms and rule to FD_+:

A5. $A \rightsquigarrow \neg\neg A$

A6. $\neg\neg A \rightsquigarrow A$

Contraposition (Con). $A \rightsquigarrow B \Rightarrow \neg B \rightsquigarrow \neg A$

Then, Routley and Meyer's *Basic Positive Logic* B_+ (cf. [15], Chap. 8] is formulated when adding to FD_+ (Trans, CI$\wedge$ and E$\vee$ are not independent) the rules MP, Suf and the following axioms and rule:

A7. $[(A \rightsquigarrow B) \wedge (A \rightsquigarrow C)] \rightsquigarrow [A \rightsquigarrow (B \wedge C)]$

A8. $[(A \rightsquigarrow C) \wedge (B \rightsquigarrow C)] \rightsquigarrow [(A \vee B) \rightsquigarrow C]$

Prefixing (Pref). $B \rightsquigarrow C \Rightarrow (A \rightsquigarrow B) \rightsquigarrow (A \rightsquigarrow C)$

Finally, the list of the basic logics is ended with Routley and Meyer's *Basic Logic* B (cf. [15], Chap. 8] that is formulated by adding A5, A6 and the rule Con to B_+.

Now, the following deep relevant extensions of B are considered (cf. [6]).

DW: B plus

A9. $(A \rightsquigarrow B) \rightsquigarrow (\neg B \rightsquigarrow \neg A)$

DJ: DW plus

$$\text{A10. } [(A \rightsquigarrow B) \wedge (B \rightsquigarrow C)] \rightsquigarrow (A \rightsquigarrow C)$$

DK: DJ plus

$$\text{A11. } A \vee \neg A$$

DR: DK plus

$$\text{Specialized reductio (sr). } A \Rightarrow \neg(A \rightsquigarrow \neg A)$$

Each of the relevant logics just defined can "deep relevantly" be supplemented with the meta-rule (see [6])

$$\text{Summation (MRs). } A \Rightarrow B \Rightarrow C \vee A \Rightarrow C \vee B$$

These deep relevant logics can be extended to the standard relevant logics as follows. In A1-A10 and MP change the deep relevant conditional $\rightsquigarrow$ for $\to$ now representing the relevant conditional. We have (some axioms and rules of DW are not independent now; see, e.g., [15]):

TW: DW plus

$$\text{A12. } (A \to B) \to [(B \to C) \to (A \to C)]$$
$$\text{A13. } (B \to C) \to [(A \to B) \to (A \to C)]$$

T: TW plus

$$\text{A14. } [A \to (A \to B)] \to (A \to B)$$
$$\text{A15. } (A \to \neg A) \to \neg A$$

E: T plus

$$\text{A16. } [[(A \to A) \wedge (B \to B)] \to C] \to C$$

R: T plus

$$\text{A17. } A \to [(A \to B) \to B]$$

RM: R plus

$$\text{A18. } A \to (A \to A)$$

TW is *Contractionless Ticket Entailment;* T, *Ticket Entailment*; E, *Entailment* (cf. [1], §26.1 concerning this axiomatization of E); R, *Logic of the Relevant Conditional*, and, finally, RM is *R-Mingle* (we remark that RM lacks the vsp: in RM the conditional $\to$ is not actually a relevant conditional).

Appendix 2. Given $\mathbf{B}_+$, (*)n and(**)n are equivalent

Firstly, we show that the

$$\text{Contraction rule (rW). } A \to (A \to B) \Rightarrow A \to B$$

and the

$$\text{Modus Ponens axiom (MPa). } [A \wedge (A \to B)] \to B$$

are equivalent. In the proofs to follow, we use $\to$ instead of $\rightsquigarrow$ since, as proved in §4, (**)n (so, MPa) cannot be a thesis of (deep relevant) logics verified by weak relevant model structures.

(1a) *MPa is derivable from* Σ *plus rW.*

We prove:

$$\text{Importation (Imp). } A \to (B \to C) \Rightarrow (A \wedge B) \to C$$

whence MPa is immediate by A1.

1. $A \to (B \to C)$	Hypothesis
2. $(A \wedge B) \to (B \to C)$	By 1 and Trans
3. $(B \to C) \to [(A \wedge B) \to C]$	By A2 and Suf
4. $(A \wedge B) \to [(A \wedge B) \to C]$	By Trans, 2, 3
5. $(A \wedge B) \to C$	By rW, 4

(1b) *rW is derivable from* Σ *plus MPa.*

Firstly, notice that the rule 'Factor'

$$\text{rF. } A \to B \Rightarrow (C \wedge A) \to (C \wedge B)$$

is immediate in Σ by A2, Trans and $\mathrm{CI}_\wedge$. Also, that the thesis 'Idempotence'

$$\text{Idem. } A \to (A \wedge A)$$

is derivable in Σ by A1, Adj and $\mathrm{CI}_\wedge$. Then, we have:

1. $A \to (A \to B)$	Hypothesis
2. $[A \wedge (A \to B)] \to B$	MPa
3. $(A \wedge A) \to [A \wedge (A \to B)]$	rF, 1
4. $(A \wedge A) \to B$	Trans, 2, 3
5. $A \to B$	Trans, 4, Idem

On the other hand, we recall that (**)n and (*)n are

$$(*)n.\ A \overset{n+1}{\to} B \Rightarrow A \overset{n}{\to} B$$

$$(**)n.\ [A \wedge (A \overset{n}{\to} B)] \Rightarrow A \overset{n}{\to} B$$

where $A \overset{n}{\to} B$ abbreviates $A \to [A \to (... \to (A \to B)...)]$ with n occurrences of A.

(2a) *(**)n is derivable from* B_+ *plus (*)n.*

$$1.\ [A \wedge (A \overset{n}{\to} B)] \to A \qquad \text{A2}$$

$$2.\ (A \to B) \to [[A \wedge (A \overset{n}{\to} B)] \to B] \qquad \text{Suf, 1}$$

By $n-1$ applications of Pref:

$$3.\ (A \overset{n}{\to} B) \to [A \overset{n-1}{\to} [A \wedge (A \overset{n}{\to} B)] \to B]] \qquad \text{(A2)}$$

Then,

$$4.\ [A \wedge (A \overset{n}{\to} B)] \to (A \overset{n}{\to} B) \qquad \text{(A2)}$$

$$5.\ [A \wedge (A \overset{n}{\to} B)] \to [A \overset{n-1}{\to} [[A \wedge (A \overset{n}{\to} B)] \to B]] \qquad \text{Trans, 3, 4}$$

On the other hand,

$$6.\ \{A \to [[A \wedge (A \overset{n}{\to} B)] \to B]\} \to$$
$$\{[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \to B]\} \qquad \text{Suf, 1}$$

whence by $n-2$ applications of Pref,

$$7.\ \{A \overset{n-1}{\to} [[A \wedge (A \overset{n}{\to} B)] \to B]\} \to$$
$$\{A \overset{n-2}{\to} [[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \to B]]\}$$

Then, by Trans, 5, 7, we have one of the two fundamental schemes in the proof:

$$8.\ [A \wedge (A \overset{n}{\to} B)] \to \{A \overset{n-2}{\to} [[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \to B]]\}$$

On the other hand,

$$9.\ \{A \to [[A \wedge (A \overset{n}{\to} B)] \overset{2}{\to} B]\} \to$$
$$\{[A \wedge (A \overset{n}{\to} B)] \to [A \wedge (A \overset{n}{\to} B)] \overset{2}{\to} B]\} \qquad \text{Suf, 1}$$

$$10.\ \{A \to [A \to [[A \wedge (A \overset{n}{\to} B)] \overset{2}{\to} B]]\} \to$$
$$\{A \to [[A \wedge (A \overset{n}{\to} B)] \overset{3}{\to} B]\} \qquad \text{Pref, 9}$$

$$11.\ \{A \to [[A \wedge (A \overset{n}{\to} B)] \overset{3}{\to} B]\} \to$$
$$[[A \wedge (A \overset{n}{\to} B)] \overset{4}{\to} B] \qquad \text{Suf, 1}$$

$$12.\ \{A \overset{2}{\to} [[A \wedge (A \overset{n}{\to} B)] \overset{2}{\to} B]\} \to$$
$$[[A \wedge (A \overset{n}{\to} B)] \overset{4}{\to} B] \qquad \text{Trans 10, 11}$$

Repeating the argumentation in lines 10-12, we get the second fundamental scheme in the proof,

$$13.\ \{A \overset{n-2}{\to} [[A \wedge (A \overset{n}{\to} B)] \overset{2}{\to} B]\} \to \{[A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B]\}$$

Then,

$$14.\ [A \wedge (A \overset{n}{\to} B)] \to [A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B] \qquad \text{Trans, 8, 13}$$

That is,

$$15.\ [A \wedge (A \overset{n}{\to} B)] \overset{n+1}{\to} B$$

whence, by (*)n,

$$16.\ [A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B$$

as was to be proved.

We remark that

$$(^{**})n.\ [A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B$$

is not derivable from B_+ and

$$(^*)n'.\ A \overset{n+2}{\to} B \Rightarrow A \overset{n+1}{\to} B$$

as shown in Appx. 3.

(2b) *(*)n is derivable from* B_+ *plus (**)n.*

1. $A \to (A \overset{n}{\to} B)$	Hypothesis
2. $(A \wedge A) \to [A \wedge (A \overset{n}{\to} B)]$	rF, 1
3. $A \to (A \wedge A)$	Idem
4. $A \to [A \wedge (A \overset{n}{\to} B)]$	Trans, 2, 3
5. $[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \overset{n-1}{\to} B]$	(**)n
6. $A \to [[A \wedge (A \overset{n}{\to} B)] \overset{n-1}{\to} B]$	(Trans, 4, 5)

On the other hand,

7. $[[A \wedge (A \overset{n}{\to} B)] \to B] \to (A \to B)$ Suf, 4
8. $\{[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \to B]\} \to$
   $[[A \wedge (A \overset{n}{\to} B)] \to (A \to B)]$ Pref, 7
9. $[[A \wedge (A \overset{n}{\to} B)] \to (A \to B)] \to [A \to (A \to B)]$ Suf, 4
10. $\{[A \wedge (A \overset{n}{\to} B)] \to [[A \wedge (A \overset{n}{\to} B)] \to B]\} \to$
    $[A \to (A \to B)]$ Trans, 8, 9
11. $[[A \wedge (A \overset{n}{\to} B)] \overset{3}{\to} B] \to [[A \wedge (A \overset{n}{\to} B)] \to$
    $[A \to (A \to B)]]$ Pref, 10
12. $\{[A \wedge (A \overset{n}{\to} B)] \to [A \to (A \to B)]\} \to$
    $[A \to [A \to (A \to B)]]$ Suf, 4
13. $[[A \wedge (A \overset{n}{\to} B)] \overset{3}{\to} B] \to (A \overset{3}{\to} B)$ Trans, 11, 12

By repeating the argumentation, we get,

14. $[[A \wedge (A \overset{n}{\to} B)] \overset{n-1}{\to} B] \to (A \overset{n-1}{\to} B)$

Then, by Trans, 6, 14,

15. $A \to (A \overset{n-1}{\to} B)$

i.e.,

16. $A \overset{n}{\to} B$

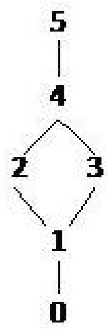
as was to be proved.

Appendix 3. Matrices

We record some matrices used in one way or another throughout the paper (designated values are starred). Save for one case (M2, below), in all matrices that follow $a \vee b$ and $a \wedge b$, for all a, $b \in K$, are understood as $max\{a, b\}$ and $min\{a, b\}$, respectively. In case a tester is needed, the reader may use that in [11]. We have tried to provide the simplest possible matrices for each one of our examples.

1. Meyer's Crystal Lattice CL

   The structure of CL is the following:



the conditional and negation being defined as follows:

$\to$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
*1	0	1	2	3	4	5	4
*2	0	0	2	0	2	5	2
*3	0	0	0	3	3	5	3
*4	0	0	0	0	1	5	1
*5	0	0	0	0	0	5	0

(We have rephrased CL in the form of the rest of matrices in the Appendix).

2. M1: (**)n is not derivable from (*)n', given B_+ (see Appx. 2).

   The schemes referred to are the following:

$$(**)n.\ [A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B$$

$$(*)n'.\ A \overset{n+2}{\to} B \Rightarrow A \overset{n+1}{\to} B$$

   M1 has the following structure:

the conditional and the negation being defined as follows:

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	1
2	1	2	3	3	2
*3	0	0	0	3	0

M1 verifies the logic B_+ (see Appx. 1) plus the thesis

$$(A \overset{4}{\rightarrow} B) \rightarrow (A \overset{3}{\rightarrow} B)$$

but falsifies ($v(A) = 2$, $v(B) = 0$)

$$[A \wedge (A \overset{2}{\rightarrow} B)] \overset{2}{\rightarrow} B$$

3. M2: a relevant logic with $[(A \wedge B) \rightarrow C] \rightarrow [A \rightarrow (B \rightarrow C)]$ as a thesis. M2 has the following structure (satisfying $\rightarrow$ and $\vee$ but not $\wedge$):

the conditional and the negation being defined as follows:

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
*1	0	1	2	3	1
*2	0	0	2	3	2
*3	0	0	2	3	0

On the other hand, $a \vee b = max\{a, b\}$ but $a \wedge b \neq inf\{a, b\}$:

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	2	2
*2	0	2	2	2
*3	0	2	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

M2 verifies all logics axiomatized with any selection of the following axioms and rules for $\rightarrow$ and $\neg$ (cf. Appx. 1).

$$
\begin{aligned}
&\text{A1. } A \rightarrow A\\
&\text{A12. } (A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]\\
&\text{A13. } (B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]\\
&\text{A14. } [A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)\\
&\text{A17. } A \rightarrow [(A \rightarrow B) \rightarrow B]\\
&\text{A5. } A \rightarrow \neg\neg A\\
&\text{A6. } \neg\neg A \rightarrow A\\
&\text{A15. } (A \rightarrow \neg A) \rightarrow \neg A
\end{aligned}
$$

And regarding conjunction and disjunction, we select, among those verified, the following:

$$
\begin{aligned}
\text{Adjunction (Adj). } & A \ \& \ B \Rightarrow A \wedge B\\
\text{Elimination of } \wedge \ (\text{E}\wedge). \ & A \wedge B \Rightarrow A, B\\
\text{A7. } & [(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]\\
\text{A3. } & A \rightarrow (A \vee B) \ / \ B \rightarrow (A \vee B)\\
\text{Elimination of } \vee \ (\text{E}\vee). \ & A \rightarrow C \ \& \ B \rightarrow C \Rightarrow (A \vee B) \rightarrow C\\
\text{A4. } & [A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]\\
\text{A19. } & (A \rightarrow B) \vee (B \rightarrow A)\\
\text{A20. } & (A \rightarrow B) \rightarrow [A \rightarrow (A \wedge B)]\\
\text{A21. } & (A \rightarrow B) \rightarrow [(A \wedge C) \rightarrow (B \wedge C)]\\
\text{A23. } & (A \rightarrow B) \rightarrow [(A \rightarrow C) \rightarrow [A \rightarrow (B \wedge C)]]
\end{aligned}
$$

and, most of all

$$\text{A24. } [(A \wedge B) \rightarrow C] \rightarrow [A \rightarrow (B \rightarrow C)]$$

Now, notice that M2 is a wr-matrix with $K_1 = \{2\}$, $K_2 = \{1\}$ and $a_F = 0$ (cf. Definition 2.5). Consequently, any logic verified by M2 has the vsp (cf. Proposition 2.8). But, on the other hand, the following facts have to be remarked:

(a) None of the logics verified by M2 includes FD_+ (see Appx. 1): A2 $(A \wedge B) \rightarrow A \ / \ (A \wedge B) \rightarrow B$ is not verified (for example, $(A \wedge B) \rightarrow A$ is falsified when $v(A) = 1$, $v(B) = 2$). In this respect, as well as in others, the logics verified by M2 are similar to those defined by Avron in [2].

(b) Any logic formulated with one of A20, A21, A23 or A24 is not included in RM (see Appx. 1): the cited theses are not theorems of RM.

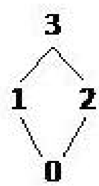
Consequently:

(c) Some of the deep relevant logics definable from M2 are not included in RM. In particular, those having A20 or A21 as axioms (A23 and A24 can be theses of no deep relevant logic).

But, most of all, we want to point out that A24 can be a thesis of such strong relevant logics as those definable from M2 as indicated above.

4. M3: strong relevant logics not including FD_+ (see Appx. 1).

   M3 is more in the spirit of standard relevant logic than M2 (it is in fact displayed in [1], §14.7). Its structure is:



   the conditional and the negation being defined as follows:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	1	0	3	1
2	0	0	2	3	2
*3	0	0	0	3	0

   M3 is a wr-matrix ($K_1 = 2$, $K_2 = 1$, $a_F = 0$). It verifies the same axioms and rules as M2 (except A3, A19-A24) and, in addition,

   A8. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
   A9. $(A \to B) \to (\neg B \to \neg A)$

   Notice that given that A2 and A3 are falsified, none of the logics verified by M3 includes FD_+ ($(A \wedge B) \to A$ and $B \to (A \vee B)$ are falsified, for example, when $v(A) = 2$ and $v(B) = 1$).

5. M4: Deep relevant logics not included in Classical Propositional Logic.

   M4 has the following structure (satisfying $\wedge$ and $\vee$ but not $\to$):

The conditional and the negation are defined as follows:

$\to$	0	1	2	3	$\neg$
0	1	1	1	1	3
*1	0	1	1	1	1
*2	0	0	2	1	2
*3	0	0	0	1	0

M3 verifies, among others, the following theses and rules: A1, A2, A3, A4, A5, A6, A7, A8, A10, A19,

A25. $[(A \to B) \wedge \neg B] \to \neg A$
A26. $(A \to B) \to [(A \vee B) \to B]$
A27. $(A \to B) \to (A \to A)$
A28. $(B \to A) \to (A \to A)$
A29. $[(A \to A) \to B] \to B$
A30. $[(A \to B) \to A] \to A$

and, most of all,

A31. $\neg(A \to B) \vee B$

which is not, of course, a classical tautology.

Now, notice that M4 is a wr-matrix (Definition 2.5). Consequently, all logics verified by M4 have the vsp (Proposition 2.8). On the other hand, it would be easy to show (but we cannot, of course, prove it here) that it is possible to build upon M4 deep relevant logics with the thesis (see [13])

A31′. $\neg(A \rightsquigarrow B) \vee B$

6. M5: Logics with the contraction rule but without the Modus Ponens axiom.

The structure of M5 is:

And the conditional and negation are defined as follows:

$\rightarrow$	0	1	2	$\neg$
0	2	2	2	2
1	0	2	2	1
*2	1	1	2	0

Following Brady's strategy in [4] for axiomatizing 3-valued and 4-valued matrices, it could be shown that M5 is axiomatized by adding the following axioms to FD (we do not try to provide a set of independent axioms):

A32. $(A \wedge \neg A) \rightarrow (B \vee \neg B)$
A33. $\neg A \rightarrow [A \vee (A \rightarrow B)]$
A34. $A \rightarrow (B \rightarrow A)$
A35. $(A \vee \neg B) \vee (A \rightarrow B)$
A36. $[(A \rightarrow B) \wedge A] \rightarrow [(\neg A \vee B) \vee \neg(A \rightarrow B)]$
A37. $[(A \rightarrow B) \wedge \neg B] \rightarrow [(\neg A \vee B) \vee \neg(A \rightarrow B)]$
A38. $\neg B \rightarrow [(A \vee \neg A) \vee \neg(A \rightarrow B)]$
A39. $[\neg(A \rightarrow B) \wedge (A \wedge \neg B)] \rightarrow [(\neg A \vee B) \vee (A \rightarrow B)]$

Let us use ΣM5 to refer to this system. Now, notice that

$$\text{A14. } [A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$$

is verified by M5, but

$$(**)n.\ [A \wedge (A \overset{n}{\rightarrow} B)] \overset{n}{\rightarrow} B$$

is falsified, for any $n \geq 1$, when A and B are assigned 2 and 0, respectively. On the other hand, we remark that, although the prefixing axiom A13 $(B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$ is verified, the rule Suf is falsified (set $v(A) = 1$, $v(B) = 2$ and $v(C) = 0$). Consequently, ΣM5 does not contain B_+. Finally, notice that A39 is not a classical tautology, but that it suffices to delete it, and then we can define from FD plus any selection of A32-A38 strong logics contained in classical logic with A14 but lacking $(**)n$.

ACKNOWLEDGEMENTS. -Work supported by research project FFI2011-28494 financed by the Spanish Ministry of Economy and Competitiviness. -G. Robles is supported by Program Ramón y Cajal of the Spanish Ministry of Economy and Competitiviness. - We thank a referee of Studia Logica for his (her) comments and suggestions on a previous draft of this paper.

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