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Blocking the routes to triviality with depth relevance

Gemma Robles and José M. Méndez

Abstract

In Rogerson and Restall's "Routes to triviality" (*Journal of Philosophical Logic*, 36, 2006, p. 435), the "class of implication formulas known to trivialize (NC)" (NC abbreviates "naïve comprehension") is recorded. The aim of this paper is to show how to invalidate any member in this class by using "weak relevant model structures". Weak relevant model structures verify deep relevant logics only.

Keywords: Naive set theory, weak relevant model structures, depth relevance, deep relevant logics.

1 Introduction

In [17], the "class of implication formulas known to trivialize (NC)" ([17], p. 435, NC abbreviates "naïve comprehension") is recorded. The aim of this paper is to show how to invalidate any member in this class by using "weak relevant model structures". Weak relevant model structures verify deep relevant logics only. Thus, it will be shown how to invalidate the elements belonging to the aforementioned class in a wide spectrum of deep relevant logics. We begin by describing the routes to triviality remarked by Rogerson and Restall. Then, we explain the notion of depth relevance introduced by Brady. Finally, we specify the structure of the paper.

1.1 Routes to triviality

In [17], languages with at least the connective $\to$ (conditional) are considered, In this paper, logic and languages are as follows:

Definition 1.1 (Languages) *The propositional language consists of a set of denumerable propositional variables and some (or all) of the following connectives:* $\to$ *(conditional),* $\wedge$ *(conjunction),* $\vee$*, (disjunction) and* $\neg$ *(negation), the biconditional (*$\leftrightarrow$*) being defined in the customary way;* A*,* B*,* C*, etc., (possibly with subscripts* $0, 1, ..., n, n+1, ...2n$*) are metalinguistic variables.*

Let us now recall how Curry's Paradox arises (cf. [9]). We set:

Definition 1.2 (Logics, logics with contraction) *The logics are formulated on the propositional languages defined in Definition 1.1 in the Hilbert-style form. A logic is a set of formulas closed under Modus Ponens (MP), Elimination of the biconditional (E$\leftrightarrow$) and uniform substitution of propositional variables. Let S be a logic. We say that S is a logic with contraction if it is closed under the contraction law (W)* $[A \to (A \to B)] \to (A \to B)$.

Consider now the Comprehension Axiom in the form $\exists y \forall x (x \in y \leftrightarrow A)$ where y is not free in A ('Naive comprehension', NC). And let S be a logic with contraction and S$'$ be a basic first order extension of S (cf. [3], p. 72). In [9] it is shown that NC trivializes S$'$ as follows:

1. $x \in x \leftrightarrow (x \in x \to p)$	By NC
2. $x \in x \to (x \in x \to p)$	E$\leftrightarrow$, 1
3. $(x \in x \to p) \to x \in x$	E$\leftrightarrow$, 1
4. $[x \in x \to (x \in x \to p)] \to (x \in x \to p)$	W
5. $x \in x \to p$	MP 2, 4
6. $x \in x$	MP 3, 5
7. p	MP 5, 6

But p is arbitrary. So, S$'$ is trivial.

Now, in [17] it is shown that a number of theses can replace W in the proof just displayed. So, we begin by describing these theses (we shall essentially maintain Rogerson and Restall's notation and terminology). Firstly, we provide a couple of auxiliary definitions:

Definition 1.3 (The classes of wffs U_ψ, V_ψ, W_ψ) *Given a propositional language, let p and q be propositional variables and $\psi(p,q)$ an arbitrary but fixed formula containing no variables other than p and q. For each $\psi(p,q)$ the classes of wffs U_ψ, V_ψ, W_ψ are defined as follows: $U_\psi = \{p, q, \psi(p,q)\}$; $V_\psi = \{p \to \psi(p,q), \psi(p,q) \to p\}$; $W_\psi = \{\psi(p,q) \to (p \to q), p \to [\psi(p,q) \to p]\}$.*

Definition 1.4 (α-formulas, β-formulas) *Let $A_1, ..., A_n, A_{n+1}$ be wffs. An α-formula is a formula of the form $A_1 \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$. And a β-formula is a formula of the form $[(...((A_{n+1} \to A_n) \to A_{n-1}) \to ...) \to A_2] \to A_1$ where $n \geq 1$.*

Next, three new classes of wffs, X_ψ, Y_ψ, Z_ψ are defined. In [17], it is proved that, under certain circumstances, members of these classes will cause the same effect caused by W when added to a basic first order extension S$'$ of a logic with contraction S.

Definition 1.5 (The classes X_ψ, Y_ψ, Z_ψ) *Let $\psi(p,q)$, U_ψ, V_ψ, W_ψ be as in Definition 1.3. Furthermore, let α_ψ be an α-formula in which $n \geq 1$, $A_i \in V_\psi$ $(1 \leq i \leq n)$ and $A_{n+1} \in U_\psi$. Then X_ψ, Y_ψ, Z_ψ are defined as follows: $X_\psi = \{\alpha_\psi : A_{n+1} = q\}$; $Y_\psi = \{\alpha_\psi : A_{n+1} = \psi(p,q)\}$; $Z_\psi = \{\alpha_\psi : A_{n+1} = p\}$.*

Notice that X_ψ, Y_ψ and Z_ψ have been defined for each wff $\psi(p,q)$. Concerning the classes of wffs just defined, the following theorem is proved in [17] (Theorem 3, p. 432).

Theorem 1.6 (Rogerson and Restall Theorem) *Let B and C be provable in a logic S. Then, NC trivializes S if anyone of the following is the case: (1) $B \in X_\psi$ (2) $B \in Y_\psi$ and $C \in W_\psi$, (3) $B \in Z_\psi$ and $C \in W_\psi$.*

(1), (2) and (3) are the routes to triviality remarked by Rogerson and Restall.

Among the members in X_ψ, Y_ψ and Z_ψ, Rogerson and Restall point out the following:

1. Members of X_ψ
   (a) The Axiom of relativity (A) $[(q \to p) \to p] \to q$ and any generalization of it (Ag) $[(...((q \to p) \to p) \to ...) \to p] \to q$ (with $n \geq 2$ occurrences of p).
   (b) The axiom of supercontraction (SW) $[p \to (p \to q)] \to q$ and any of its generalizations (SWg) $[p \to (p \to (... \to (p \to q)...)] \to q$ (with $n \geq 2$ occurrences of p).
   (c) The axiom L $[p \to (p \to q)] \to [[(p \to q) \to p] \to q]$ as well as any extension of it obtained by adding any number n of antecedents $A_1, ..., A_n$ (Lg) $A_1 \to [A_2 \to (... \to (A_n \to [[p \to (p \to q)] \to [[(p \to q) \to p] \to q])...)]$ (with A_i $(1 \leq i \leq n)$ either of the form $p \to (p \to q)$ or $(p \to q) \to p$).
2. Members of Y_ψ
   (a) The Axiom of contraction (W) $[p \to (p \to q)] \to (p \to q)$ and any of its generalizations (Wg) $\{p \to [p \to (... \to (p \to q)...)]\} \to \{[p \to (... \to (p \to q)...)]\}$ (with $n+1$ occurrences of p in the antecedent of Wg and n in the consequent).
3. Members in Z_ψ
   (a) Peirce's Law (PL) $[(p \to q) \to p] \to p$ and any of its extensions (PLg) $A_1 \to [A_2 \to (... \to (A_n \to [(p \to q) \to p] \to p])...)]$ where A_i $(1 \leq i \leq n)$ is either of the form $p \to (p \to q)$ or $(p \to q) \to p)$, as in the case of L.

In order to describe a first approximation to the aims of the paper, let us illustrate with an example how Curry's Paradox can be derived by following Rogerson and Restall's routes (for instance, we take route (3)). Let S and S$'$ be as in the proof of Curry's Paradox displayed above, and consider the wffs

1. $[\psi(p,q) \to p] \to [[p \to \psi(p,q)] \to p]$

and

$$2.\ \psi(p,q) \to (p \to q)$$

where (1) is a member of Z_ψ ($n = 2$) and (2) is a member of W_ψ. Then, substitute $a \in a$ for p in the wffs $\psi(p,q)$, (1) and (2), and let $a = \{x : \psi(x \in x, q)\}$. We have:

3. $\psi(a \in a, q) \to a \in a$	NC, E$\leftrightarrow$
4. $a \in a \to \psi(a \in a, q)$	NC, E$\leftrightarrow$
5. $[\psi(a \in a, q) \to a \in a] \to [[a \in a \to \psi(a \in a, q)] \to a \in a]$	1
6. $a \in a$	Mp (twice), 3, 4, 5
7. $\psi(a \in a, q)$	MP 4, 6
8. $\psi(a \in a, q) \to (a \in a \to q)$	2
9. q	MP (twice) 6, 7, 8

The aim of this paper is to prove that if $\psi(p,q)$ is a positive wff, then routes to triviality described in Rogerson and Restall's theorem are blocked in any logic verified by weak relevant model structures of a certain type. In particular, route (1) is blocked whenever $\psi(p,q)$ is any wff subject to the conditions in Definition 1.3 (it need not be a positive formula). The notion of a "positive formula" and a "weak positive formula" are defined as follows.

Definition 1.7 (Positive and weak positive wffs) *Let L be a propositional language (cf. Definition 1.1). Then, (1) a positive formula is a wff in which* $\neg$ *does not appear; (2) a weak positive formula is a wff in which* $\to$ *and* $\neg$ *do not appear.*

Before explaining the notion of depth relevance and related notions, let us note a couple of remarks concerning the definitions of the classes X_ψ, Y_ψ and Z_ψ.

Rogerson and Restall's definition differs from Definition 1.5 in the two following respects: (1) The case $n = 0$ is allowed. (2) A_i ($1 \leq i \leq n$) can be a tautology of S.

We then note the following remarks concerning points (1) and (2):

1. If $\alpha_\psi \in X_\psi$ or $\alpha \in Z_\psi$ and $n = 0$, then it is obvious that α_ψ immediately trivializes S.

2. If $\alpha_\psi \in Y_\psi$ and $n = 0$, α_ψ cannot generally be invalidated, given that $p \to (p \vee q)$, $(p \wedge q) \to p$ or $(p \to q) \to (p \to q)$ are wffs comprehended in the general form $\psi(p,q)$. Moreover, these wffs are theorems in all deep relevant logics defined thus far (cf. Appendix 1). So, they do not cause Curry's Paradox 'per se', i.e., by themselves. But, on the other hand, and leaving aside Curry's Paradox, it certainly seems difficult to find grounds to object to some thesis as $(p \to q) \to (p \to q)$, for example. Consequently, the case in which $\alpha_\psi \in Y_\psi$ and $n = 0$ cannot in general be considered here (anyway, notice that members in Y_ψ require the help of some element in W_ψ in order to cause Curry's Paradox according to route (2)).

3. Obviously, the case in which A_i $(1 \leq i \leq n)$ is a tautology cannot be treated unless the structure of A_i is specified. Thus, this case is not treated in this paper either.

1.2 Depth relevance

The "depth relevance condition" (drc) is introduced in [3]. In the referred paper, the drc is motivated as a necessary condition (stated in syntactic terms) for some paraconsistent logics rejecting the Contraction Law (W), which is used in deriving Curry's Paradox, as it has been shown above. Brady actually proceeds as follows. He labels a Curry's Paradox 'basic' if it is derived by using the Contraction rule $(A \to (A \to B) \Rightarrow A \to B)$. Then, he remarks: "Indeed, the depth relevance condition provides a non-*ad hoc* way of avoiding such paradoxes because it is not specifically aimed at avoiding the basic Curry-paradoxes and it does have some alternative intuitive appeal, as indicated earlier in the paper." ([3], p. 72) The present paper provides additional support for the drc in the sense of Brady's remark, by showing that depth relevance can be employed in avoiding not only basic Curry-paradoxes but also those more complex types described in Rogerson and Restall's theorem.

The aim of [3] is to define the main logic with the drc. Brady's strategy consists in restricting the class of logics with the variable-sharing property (vsp, see Definition 1.8 below) verified by Meyer's Crystal matrix CL (see Appendix 2 where CL is displayed) with the drc. Then, he chooses the logic DR as the preferred one among those defined from CL as indicated above (see Appendix 1 for a definition of DR and other relevant and deep relevant logics). Brady's investigations on the topic have been pursued in [4], [5] and most of all in [7]. And we remark that currently Brady prefers to found his logics for naïve set theory in his semantics of "meaning containment" than to found them on the drc (cf. [5] and [7]). We also remark that DJ is Brady's preferred logic in these subsequent works (see [7], [8]).

On the other hand, Brady's strategy has been generalized in [15]. We have shown how to define a class of deep relevant logics from each weak relevant matrix. "Weak relevant matrices" are defined in [13], where they are characterized as matrices verifying only logics with the "variable-sharing property". Finally, in [14] weak relevant model structures defined on weak relevant matrices are used in order to invalidate the *Generalized Modus Ponens Axiom* and other related theses and rules.

But let us define the notion of "depth relevance". As it is known, the following is a necessary property of any relevant logic S, according to Anderson and Belnap, (cf. [1]):

Definition 1.8 (Variable-sharing property —vsp) *If $A \to B$ is a theorem of S, then A and B share at least one propositional variable.*

As it has been remarked, in [3], Brady strengthens the vsp to the drc. In order to define the latter, we need the notion of "depth of a subformula within a formula" (cf. [3], [6]] §11).

Definition 1.9 (Depth of a subformula within a formula) *Let A be a wff and B a subformula of A. Then, "the depth of B in A" (in symbols, $d[B, A]$) is inductively defined as follows: (1) B is A. Then, $d[B, A] = 0$. (2) B is $\neg C$. Then, $d[C, A] = n$ if $d[\neg C, A] = n$. (3) B is $C \wedge D$ $(C \vee D)$. Then, $d[C, A] = d[D, A] = n$ if $d[C \wedge D, A] = n$ $(d[C \vee D, A] = n)$. (3) B is $C \rightarrow D$. Then, $d[C, A] = d[D, A] = n + 1$ if $d[C \rightarrow D, A] = n$.*

Then, we set:

Definition 1.10 (Depth relevance condition —drc) *Let S be a propositional logic with $\rightarrow$ among its connectives (cf. Definition 1.1). S has the depth relevance condition (or S is a deep relevant logic) if in all theorems of S of the form $A \rightarrow B$ there is at least a propositional variable p common to A and B such that $d[p, A] = d[p, B]$.*

Regarding the notions just defined, we shall employ the notation recorded in the following remark,

Remark 1.11 (Notation) *Let A, B be wffs, C and D subformulas of A, E a subformula of C and p a propositional variable. Then, $\overset{n}{p}(A)$ means "the $n-th$ occurrence (from left to right) of p in A". Next, $\overset{o}{p}(A)$ means "the last occurrence (from left to right) of p in A". And, $\overset{n}{p}(A(C))$ can be read as "the $n-th$ occurrence (from left to right) of p in the subformula C of A"; and thus, $\overset{n}{p}(A(C(E)))$ or $\overset{o}{p}(A(C(E)))$ can be read similarly. On the other hand, $d[p, A(C)] = s$ means the "depth of p (appearing in C) in A is s" and $d[p, A(C(E))] = s$ can be read similarly. So, $d[\overset{n}{p}, A] = s$ means "the depth of the $n-th$ occurrence (from left to right) of p in A is s", and $d[\overset{n}{p}, A(C)] = s$ or $d[\overset{o}{p}, A(C(E))] = s$ can be read similarly. Finally, $\Delta(A, B) = \emptyset$ means that A and B do not share variables at the same depth.*

Example 1.12 (Depth. depth relevance) *Consider the following wffs: (1) $(p \rightarrow \neg q) \rightarrow [(\neg r \wedge s) \rightarrow [(t \vee u) \rightarrow w]]$; (2) $(p \rightarrow q) \rightarrow [[p \rightarrow (q \rightarrow r)] \rightarrow (p \rightarrow r)]$ and (3) $[\underline{p} \rightarrow (p \rightarrow q)] \rightarrow (\underline{p} \rightarrow q)$*

We have: (a) the variables p, q, r and s have depth 2 in (1); the variables t, u and w have depth 3 in (1); (b) antecedent and consequent of (3) have the underlined p at the same depth (notice that (3) is an instance of the Contraction Law W); (c) antecedent and consequent of (2) do not share variables at the same depth; that is, $\Delta(p \rightarrow q, [(p \rightarrow (q \rightarrow r)] \rightarrow (p \rightarrow r)) = \emptyset$; (d) let us refer by 2 to the wff in (2). Then, $d[\overset{1}{q}, 2] = 2$, $d[\overset{2}{q}, 2] = 4$ (or, equivalently, $d[\overset{o}{q}, 2] = 4$); $d[r, 2(p \rightarrow r)] = 3$, $d[r, 2(p \rightarrow (q \rightarrow r)(q \rightarrow r))] = 4$.

This introduction is ended by displaying the structure of the paper. In Section 2 ("Basic weak relevant model structures"), it is shown how to define basic weak relevant model structures (wr-ms) on weak relevant matrices (wr-matrices). Then, the fundamental theorem on wr-ms in proved: if $A \rightarrow B$ is valid in a wr-ms, then $\Delta(A, B) \neq \emptyset$. This theorem is a generalization (for basic wr-ms) of Brady's theorem stating the same fact for the logic DR that is verified by the wr-ms $\mathcal{M}_{\mathrm{CL}}$ built upon Meyer's matrix CL (see Theorem 3 in [3]). In Section 3 ("Other weak relevant modal structures"), by strengthening the conditions on wr-matrices, other types of wr-matrices are introduced. Then, corresponding types of wr-ms built upon the wr-matrices are defined. Finally, generalization and extensions of the fundamental theorem in wr-ms are proved for these new wr-ms. In Section 4 ("Blocking the routes to triviality"), it is shown that the theorems proved in Section 3 invalidate general classes of wffs among which those defined in Rogerson and Restall's theorem are to be found. These theorems are now used in Section 4 to prove a series of facts whence it follows that the routes to triviality remarked by Rogerson and Restall are blocked in certain natural wr-ms. In Section 5 ("Concluding remarks"), we draw some conclusions from the results obtained and suggest some directions for further work in the same line. We have added two appendices. In Appendix 1, the main relevant and deep relevant logics mentioned throughout the paper (and in Appendix 2) are defined. In Appendix 2, the notion of a logical matrix and related notions are, for definiteness, revisited. Then, examples of the different types of wr-matrices are displayed. Results of [3], [14] and [15] are not presupposed.

2 Basic weak relevant model structures

In this section, "weak relevant model structures" (wr-ms, for short) are defined and the fundamental theorem on wr-ms is proved.

2.1 Basic wr-ms

We begin by defining the notion of a "weak relevant matrix" (cf. the definition of "logical matrix" in Appendix 2).

Definition 2.1 (Weak relevant matrices —wr-matrices) *Let M be a logical matrix, K_1 and K_2 be non empty subsets of K such that $K_1 \cap K_2 = \emptyset$ and $a_F \in F$. Finally, the following conditions are fulfilled: (1) $\forall x \forall y \in K_1$ $f_\wedge(x, y)$ & $f_\vee(x, y)$ & $f_\to(x, y)$ & $f_\neg(x) \in K_1$; (2) $\forall x \forall y \in K_2$ $f_\wedge(x, y)$ & $f_\vee(x, y)$ & $f_\to(x, y)$ & $f_\neg(x) \in K_2$; (3) $\forall x \in K_1 \forall y \in K_2$ $f_\to(x, y) = a_F$; (4) $\forall x \in K_1 \cup K_2$ $f_\to(x, a_F) = a_F$. Then, M is a weak relevant matrix (wr-matrix, for short).*

Example 2.2 (Some wr-matrices) *All matrices displayed in Appendix 2 are wr-matrices. For instance, Meyer's matrix M_{CL} ($M1$) is wr-matrix where $K_1 = \{2\}$, $K_2 = \{3\}$ and $a_F = 0$.*

Concerning wr-matrices, we have the following basic proposition (cf. Proposition 2.8 in [14]).

Proposition 2.3 (Logics verified by wr-matrices have the vsp) *Let M be a wr-matrix and S be a logic verified by it. Then, S has the vsp.*

Proof. Assume the hypothesis of Proposition 2.3 and let $A \to B$ be a wff where A and B do not share propositional variables. We show that $A \to B$ is falsified by M. Let $a \in K_1$ and $b \in K_2$. Define an assignment on M, v, such that for each propositional variable p in A (respectively, in B), $v(p) = a$ (respectively, $v(p) = b$). By induction on the length of A and B and conditions 1, 2 in Definition 2.1, it follows that $v(A) = a$ and $v(B) = b$. Then, by condition 3 (Definition 2.1), $v(A \to B) = a_F$. ■

Example 2.4 (Some logics verified by wr-matrices) *In Appendix 2, we have recorded some wr-matrices and a number of logics verified by each one of them. Notice that all these logics have the vsp. That is, all are relevant logics in the minimal sense of the term.*

In what follows, our aim is to show how to define a class of deep relevant logics from each wr-matrix. In order to do this, "wr-model structures" and "valuation on model structures" are defined. We follow Brady's strategy in [3] where he shows how to define a class of deep relevant logics from Meyer's matrix M_{CL} (cf. $M1$ in Appendix 2). We have maintained his notation and terminology as much as possible.

Definition 2.5 (Wr-model structures —wr-ms) *Let M be a wr-matrix. A wr-model structure (wr-ms for short), $\mathcal{M}_M$, is the set $\{M_0, M_1, ..., M_n, ..., M_\omega\}$ where $M_0, M_1, ..., M_n, ..., M_\omega$ are all identical matrices to the wr-matrix M.*

Now, before defining valuations and interpretations in wr-ms, it is important to distinguish the connective defined by the function $f_\to$ in the wr-matrix from the conditional of the logical language (cf. Definition 1.1). The former shall be denoted by $\overset{M}{\to}$, where the label refers to the matrix M. In our papers preceding this one, we considered logical languages in which a relevant and a deep relevant conditional could be present. The former was interpreted according to the wr-matrix M. However, in the present paper, we shall concentrate on the deep relevant conditional, which is the only one considered.

Definition 2.6 (Valuations and interpretations in wr-ms) *Let $\mathcal{M}_M$ be a wr-ms. By v_i it is designated a function from the set of all propositional variables to K in M_i $(0 \leq i \leq \omega)$. Then, a valuation v on $\mathcal{M}_M$ is a set of functions v_i for each $i \in \{0, 1, ..., n, ..., \omega\}$. Given a valuation v, each v_i is extended to an interpretation I_i of all wffs according to the following conditions. For each propositional variable p, and wffs A, B: (i) $I_i(p) = v_i(p)$; (ii) $I_i(\neg A) = \neg I_i(A)$; (iii) $I_i(A \wedge B) = I_i(A) \wedge I_i(B)$; (iv) $I_i(A \vee B) = I_i(A) \vee I_i(B)$; (v) $I_i(A \overset{M}{\to} B) = I_i(A) \overset{M}{\to} I_i(B)$ where (i)-(v) are calculated according to the wr-matrix M. In addition, formulas of the form $A \to B$ are evaluated as follows ($a \in T$. Cf. Definition 2.1): (via) $i = 0$: $I_i(A \to B) = a$; (vib) $0 < i < \omega$: $I_i(A \to B) = I_{i-1}(A \overset{M}{\to} B)$; (vic) $i = \omega$: $I_\omega(A \to B) \in T$ iff $I_j(A \overset{M}{\to} B) \in T$ for all j $(0 \leq j \leq \omega)$. Then, the interpretation I on $\mathcal{M}_M$ extending v is the set of functions I_i for each $i \in \{0, 1, ..., n, ..., \omega\}$.*

Next, validity is defined as follows.

Definition 2.7 (Validity in wr-ms) *Let $\mathcal{M}_M$ be a wr-ms and $B_{1,...,}B_n, A$ wffs. A is valid in $\mathcal{M}_M$ (in symbols $\models_{\mathcal{M}_M} A$) iff $I_\omega(A) \in T$ for all valuations v. And the rule $B_{1,...,}B_n \Rightarrow A$ preserves $\mathcal{M}_M$-validity iff, if $I_\omega(B_1) \in T, ..., I_\omega(B_n) \in T$, then $I_\omega(A) \in T$ for all valuations v.*

This definition is extended to the case of propositional logics:

Definition 2.8 (Logics verified by wr-ms) *Let $\mathcal{M}_M$ a wr-ms and S be a logic (cf. Definition 1.2). $\mathcal{M}_M$ verifies S iff all axioms of S are $\mathcal{M}_M$-valid and all rules of S preserve $\mathcal{M}_M$-validity.*

Example 2.9 (Some wr-ms) *In Appendix 2, some wr-matrices are recorded. On each one of these wr-matrices, a wr-ms can be defined as indicated in Definition 2.5, validity being understood as in Definition 2.7. For instance, the wr-ms $\mathcal{M}_{CL}$ is defined on the matrix CL (M1, cf. Appendix 2). $\mathcal{M}_{CL}$ verifies DR and other deep relevant logics (cf. [3] and Appendix 1). Or, to take another example, the wr-ms $\mathcal{M}_{M_{SUM}}$ is built upon the matrix M_{SUM} (M4, cf. Appendix 2) which verifies Routley and Meyer's basic logic B plus the axiom "summation" (a_{25} in Appendix 1 (cf. [15]).*

2.2 The fundamental theorem on wr-ms

Next, we prove a lemma leaning on which the fundamental theorem on wr-ms is proved. This lemma will also be useful in the following section. Firstly, we define the notion of "degree of a formula" (in symbols, $deg(A)$ for a wff A) inductively as follows.

Definition 2.10 (Degree of formulas) *(1) If A is a propositional variable, then $deg(A) = 0$. (2) If A is of the form $\neg B$ and $deg(B) = n$, then $deg(A) = n$. (3) If A is of the form $B \vee C$ or $B \wedge C$, $deg(B) = n$ and $deg(C) = m$, then $deg(A) = max\{m, n\}$. (4) If A is of the form $B \to C$, $deg(B) = n$ and $deg(C) = m$, then $deg(A) = max\{m, n\} + 1$.*

Therefore, the degree of a formula A is the maximum number of nested '$\to$'s in A.

The lemma that follows (the lemma supporting the fundamental theorem) shows how to assign some elements of K_1 (or K_2) to any wff A.

Lemma 2.11 (Interpreting wffs with K_1 and K_2) *Let $\mathcal{M}_M$ be a wr-ms where $a \in K_1$ and $b \in K_2$; and let A be a wff of degree r. Then, for all interpretations I, I′ defined on $\mathcal{M}_M$ as indicated below, we prove, for each subformula B of A and for each depth d that B occurs in A: (1)$I_{r-d}(B) \in K_1$ (in particular, $I_r(A) \in K_1$). (2) $I'_{r-d}(B) \in K_2$ (in particular, $I'_r(A) \in K_2$).*

Now, I and I′ are defined by extending the valuations v, v′, respectively, according to clauses (i)-(vi) (Definition 2.6). These valuations are in their turn *defined as follows. For each propositional variable p, we set: (1) $v_{r-d}(p) = a$ for each depth d that p occurs in A; (2) $v'_{r-d}(p) = b$ for each depth d that p occurs in A; (3) $v_j(p)$ and $v'_j(p)$ are arbitrarily assigned if $j > r$ or $j = r - d$ but p does not occur at depth d in A.*

Proof. We prove case (1) (the proof of case (2) is similar). Let $c \in K$ and $v_j(p) = v'(p) = c$ if $j > r$ or $j = r - d$ but p does not occur at depth d in A. Then, notice that for each $i \in \{0, 1, ..., n..., \omega\}$ v_i and v'_i have been defined. Now, the proof is by induction on the evaluation procedure of subformulas of A (cf. Definition 2.1 and Definition 2.6).

1. B is a propositional variable. Then, case (1) follows by definition of v and (i) (Definition 2.6).

2. B is of the forms $\neg C$, $C \wedge D$ or $C \vee D$. Then, the proof is immediate by condition (1) (Definition 2.1) and (ii)-(iv) (Definition 2.6). Let us, for instance, consider the case when B is of the form $C \vee D$. If $d[C \vee D, A] = d$, then $d[C, A] = d[D, A] = d$. By hypothesis of induction, $I_{r-d}(C) \in K_1$ and $I_{r-d}(D) \in K_1$, whence $I_{r-d}(C \vee D) \in K_1$ by condition (1) (Definition 2.1) and (iv) (Definition 2.6).

3. B is of the form $C \rightarrow D$. If $d[C \rightarrow D, A] = d$, then $d[C, A] = d[D, A] = d+1$. By hypothesis of induction, $I_{r-(d+1)}(C) \in K_1$ and $I_{r-(d+1)}(D) \in K_1$, that is, $I_{r-(d+1)}(C \overset{M}{\rightarrow} D) \in K_1$ by condition (1) (Definition 2.1) and (v) (Definition 2.6). So, $I_{r-d}(C \rightarrow D) \in K_1$ by clause (vib) (Definition 2.6).

With the proof of subcase 3, case (1) is proved. Then, in particular, $I_r(A) \in K_1$ as A is a subformula of itself at depth 0 in A. ■

Theorem 2.12 (The fundamental theorem on basic wr-ms) *Let $\mathcal{M}_M$ be a wr-ms and suppose that A and B are wffs such that $\vDash_{\mathcal{M}_M} A \rightarrow B$. Then, $\Delta(A, B) \neq \emptyset$ (that is, A and B share at least one variable at the same depth, cf. Remark 1.11).*

Proof. For reductio, suppose that $A \rightarrow B$ is a wff such that A and B do not share variables at the same depth. Now, let $\mathcal{M}_M$ be a wr-ms where $a \in K_1$ and $b \in K_2$. We show that $A \rightarrow B$ is not $\mathcal{M}_M$-valid. Let $deg(A \rightarrow B) = r$. Then notice that $deg(A) \leq r - 1$ and $deg(B) \leq r - 1$ and either $deg(A) = r - 1$ or $deg(B) = r - 1$. So, the maximum depth of a subformula occurring in either A or B is $r - 1$. Then, the following valuation v is defined. For each propositional variable p, we set: (1) $v_{r-d-1}(p) = a$ for each depth d that p occurs in A. (2) $v_{r-d-1}(p) = b$ for each depth d that p occurs in B. (3) $v_j(p)$ is arbitrarily assigned if $j \geq r$ or $j = r - d - 1$ but p occurs at depth d neither in A nor in B. Next, v is extended to an interpretation I according to clauses (i)-(vi) (Definition 2.6). By Lemma 2.11, we have:

1. For each subformula C of A and for each depth d that C occurs in A, $I_{r-d-1}(C) \in K_1$.

2. For each subformula C of B and for each depth d that C occurs in B, $I_{r-d-1}(C) \in K_2$.

Now, as A (respectively, B) is a subformula of itself at depth 0 in A (respectively, in B), we have $I_{r-1}(A) \in K_1$ and $I_{r-1}(B) \in K_2$. By condition (3) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-1}(A \overset{M}{\to} B) = a_F$ whence by clause (vib) (Definition 2.6), $I_r(A \to B) = a_F$, i.e., $I_\omega(A \to B) \notin T$ (clause (vic), Definition 2.6), as it was to be proved. ■

Theorem 2.12 is a generalization (to any wr-ms) of Brady's Theorem 1 ([3], pp. 68-69] proved for the wr-ms $\mathcal{M}_{\mathrm{CL}}$ (cf. Example 2.9).

Consider now t1-t10 below. Although each one of them can appear in logics with the vsp, these are theses containing instances in which the antecedent and consequent do not share variables at the same depth. (Save for t6 and t7, these wffs are theorems of relevant logic R —see [1] and Appendix 1. Concerning t6 and t7 see [13].)

t1. $(A \to B) \to [(B \to C) \to (A \to C)]$

t2. $(B \to C) \to [(A \to B) \to (A \to C)]$

t3. $[(A \to A) \to B] \to B$

t4. $[A \wedge (A \to B)] \to B$

t5. $A \to [(A \to B) \to B]$

t6. $A \to (A \to A)$

t7. $[(A \to B) \to A] \to A$

t8. $(A \to \neg A) \to \neg A$

t9. $[(A \to B) \wedge \neg B] \to \neg A$

t10. $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

Now, Theorem 2.12 guarantees that t1-t10 and similar formulas are ruled out by any wr-ms (cf. Example 4.1 in [14]).

Remark 2.13 *Consider, however, the following wffs where metalinguistic variables that can share propositional variables (when substituted by particular wffs) at the same depth are underlined (double underlined).*

t11. $[\underline{A} \to (A \to B)] \to (\underline{A} \to B)$

t12. $[\underline{A} \to (B \to C)] \to [(\underline{A} \wedge B) \to C]$

t13. $(\underline{A} \to B) \to [\underline{A} \to (A \to B)]$

t14. $(A \to \underline{B}) \to [\underline{B} \to (A \to B)]$

t15. $[(\underline{A} \to \underline{\underline{B}}) \to A] \to [(\underline{A} \to \underline{\underline{B}}) \to B]$

t16. $[(\underline{B} \to \underline{\underline{A}}) \to A] \to [(\underline{\underline{A}} \to \underline{B}) \to B]$

t17. $[(\underline{A} \wedge B) \to C] \to [\underline{A} \to (B \to C)]$

As pointed out in [14], the problem with t11-t17 is that they break the "depth relevant condition" (drc) when added to weak positive logics with this property.

Nevertheless, t11-t17 are not falsified by Theorem 2.12 which only falsify wffs of the form $A \to B$ when A and B do not share propositional variables at the same depth.

The aim of the following section is (a) to prove a generalization of Theorem 2.12 leaning on which t11-t17 are falsified along with many other wffs inacceptable from the deep relevant point of view, and (b) to define more restricted classes of wr-ms that falsify undesirable wffs that cannot be falsified by the aforementioned generalization of Theorem 2.12.

3 Other weak relevant model structures

In some sense, this is the main section of the paper. Its aims are (1) to define new types of wr-matrices by strengthening the conditions on wr-matrices as defined in the preceding section; (2) to define the types of wr-ms corresponding to the new types of matrices defined in (1); and finally, (3) to prove some generalizations and extensions of the fundamental theorem for basic wr-ms proved in Section 2 (Theorem 2.12). These generalizations and extensions of the fundamental theorem will be used for blocking the routes to triviality described in Rogerson and Restall's theorem (Theorem 1.6) in the following section.

3.1 Other wr-ms

We begin by defining new types of matrices.

Definition 3.1 (Weak relevant matrices. Type 1) *Let M be a wr-matrix such that $a_T \in T$. Furthermore, in addition to conditions (1)-(4) in Definition 2.1, the following conditions are fulfilled: (5) $f_\to(a_T, a_F) = a_F$; (6) $\forall x \in K_2$ $f_\to(a_T, x) = a_F$; 7. $f_\wedge(a_F, a_F) = f_\vee(a_F, a_F) = a_F$; (8) $\forall x \in K_2$ $f_\to(a_F, x) = a_T$. Then, M is a weak relevant matrix of type 1, wr(1)-matrix for short.*

Example 3.2 (Some wr(1)-matrices) *The following matrices in Appendix 2 are wr(1)-matrices: $M1, M2, M3, M7$ and $M8$.*

Definition 3.3 (Weak relevant matrices. Type 2) *Let M be a wr-matrix fulfilling the following conditions (in addition to (1)-(4) in Definition 2.1 and (5), (6), (7) in Definition 3.1): (9) $f_\wedge(a_T, a_T) = f_\vee(a_T, a_T) = f_\to(a_T, a_T) = a_T$; (10) $\forall x \in K_2$ $f_\to(x, a_T) = a_T$. Then, M is a weak relevant matrix of type 2, wr(2)matrix for short.*

Example 3.4 (Some wr(2)-matrices) *The following matrices in Appendix 2 are wr(2)m-matrices: $M1, M2, M4, M6$ and $M8$.*

Definition 3.5 (Weak relevant matrices. Type 1, 2) *A weak relevant matrix of type 1, 2, a wr(1, 2)-matrix, for short, is a wr-matrix such that $a_T \in T$ and fulfills (in addition to conditions (1)-(4) in Definition 2.1), conditions (5)-(10) in Definition 3.1 and Definition 3.3.*

Example 3.6 (Some wr(1, 2)-matrices) *The following matrices in Appendix 2 are wr(1, 2)-matrices:* $M1, M2$ *and* $M8$.

Now, the new types of weak relevant model structures are defined in a similar way to which wr-ms were defined from wr-matrices (cf. Definition 2.5).

Definition 3.7 (wr(1)-ms, wr(2)-ms and wr(1, 2)-ms) *Let* M *be a wr(1)-matrix. A wr(1)-model structure (wr(1)-ms, for short) is the set* $\{M_0, M_1, ..., M_n, ..., M_\omega\}$ *where* $M_0, M_1, ..., M_n, ..., M_\omega$ *are all identical matrices to* M. *Then, wr(2)-model structures (wr(2)-ms) and wr(1, 2)-model structures (wr(1,2)-ms) are defined similarly (cf. Definition 2.5).*

Example 3.8 (Some wr(1)-ms, wr(2)-ms and wr(1, 2)-ms) *A wr(1)-ms can be defined on each one the wr(1)-matrices in Appendix 2 as indicated in Definition 3.7. And particular wr(2)-ms and wr(1, 2)-ms are obtained similarly. Consider, for instance, the wr-model structures in Example 2.9:* $\mathcal{M}_{CL}$ *is a wr(1, 2)-ms and* $\mathcal{M}_{MSUM}$ *is a wr(2)-ms (*$\mathcal{M}_{MSUM}$ *is not a wr(1)-ms).*

Remark 3.9 (Valuations, interpretations, validity) *Given that wr(1)-ms, wr(2)-ms and wr(1, 2)-ms are weak relevant model structures, we remark that valuations, interpretations and validity in the new wr-ms are understood exactly as in Definition 2.6 and Definition 2.7. Also, the notion of a "logic verified by a wr(1)-ms (wr(2)-ms, wr(1, 2)-ms)" is understood according to Definition 2.8.*

We note that in order to fulfill the aims of the paper, stated in Section 1, there is a type of wr-matrices (and, so, of wr-ms) yet to be defined (in Section 3.4 below). But for now, we shall proceed into proving the generalizations of Theorem 2.12 referred to above. The wr-ms defined thus far (wr(1)-ms, wr(2)-ms and wr(1, 2)-ms) provide the most useful (i.e. generally applicable) generalizations and extensions of Theorem 2.12.

3.2 The main property of α-wffs in wr-ms

In the rest of this section, we shall essentially be dealing with α-formulas and β-formulas. In this subsection, we first prove a preliminary property and then, the main property of α-wffs. We begin by recalling the definition of this type of formulas (cf. Definition 1.4). An α-formula is a formula of the form $A_1 \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ and a β-formula is a formula of the form $[(...((A_{n+1} \to A_n) \to A_{n-1}) \to ...) \to A_2] \to A_1$ where $n \geq 1$ and $A_1, ..., A_{n+1}$ are wffs (cf. Definition 1.1 on the languages used in the paper).

Next, we record a remark on degree and depth in α-wffs and β-wffs.

Remark 3.10 (Degree and depth in α-wffs and β-wffs) *Let* Θ *be an* α*-formula or a* β*-formula of degree* r. *We note the following facts (cf. Remark 1.11 on notation, Definition 1.9 on the notion of "depth" and Definition 2.10 on that of "degree"):*

1. *For each* A_k $(1 \leq k \leq n)$, $d[A_k, \Theta] = k$.

2. $d[A_{n+1}, \Theta] = n$

3. $deg(A_{n+1}) \leq r - n$

4. *For each* A_k $(1 \leq k \leq n)$*,* $deg(A_k) \leq r - k$*.*

5. *Let* p *be a variable in* A_k $(1 \leq k \leq n)$*. (a) If* $deg(A_k) = 0$*, then* $d[p, A_k] = 0$ *and* $d[p, \Theta(A_k)] = k$*. (b) If* $deg(A_k) = s$ $(s \geq 1)$*, then* $d[p, A_k] = m$ $(1 \leq m \leq s)$ *and* $d[p, \Theta(A_k)] = j$ $(k + 1 \leq j \leq k + s)$*.*

6. *Let* p *be a variable in* A_{n+1}*. Then: (a) If* $deg(A_{n+1}) = 0$*, then* $d[p, A_{n+1}] = 0$ *and* $d[p, \Theta(A_{n+1})] = n$*. (b) If* $deg(A_{n+1}) = s$ $(s \geq 1)$*, then* $d[p, A_{n+1}] = m$ $(1 \leq m \leq s)$ *and* $d[p, \Theta(A_{n+1})] = j$ $(n + 1 \leq j \leq n + m)$*.*

The preliminary property which we referred to above is the following.

Lemma 3.11 (A preliminary property of α-wffs in wr-ms) *Let* $\mathcal{M}_M$ *be a wr-ms and* (Θ) $A_1 \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ *be a* $\mathcal{M}_M$*-valid* α*-formula. Then,* $\Delta(\Theta(A_{n+1}), \Theta(A_k)) \neq \emptyset$ *for some* A_k $(i \leq k \leq n)$ *(i.e., there is some propositional variable* p *in* A_{n+1} *such that* $d[p, \Theta(A_{n+1})] = d[p, \Theta(A_k)]$ *for some* A_k $(1 \leq k \leq n)$*).*

Proof. Assume the hypothesis of Lemma 3.11. (1) $n = 1$. The proof follows by Theorem 2.12. (2) $n > 1$. For reductio, suppose $\Delta(\Theta(A_{n+1}), \Theta(A_k)) = \emptyset$ for each A_k $(i \leq k \leq n)$. Then, we prove that Θ is not $\mathcal{M}_M$-valid, which contradicts the hypothesis.

Let $a \in K_1$, $b \in K_2$, $c \in K$ and $deg(\Theta) = r$. We define a valuation v as follows. For each propositional variable p, we set: (1) $v_{r-d-k}(p) = a$ for each depth d that p occurs in each A_k $(1 \leq k \leq n)$; (2) $v_{r-d-n}(p) = b$ for each depth d that p occurs in A_{n+1}; (3) $v_j(p) = c$ if $j \geq r$ or $j = r - d - l$ $(1 \leq l \leq n + 1)$ but p occurs at depth d in no A_l $(1 \leq l \leq n + 1)$. Notice that v_i has been defined for each $i \in \{1, 2, ..., n, ..., \omega\}$, and that v is consistent since (by the reductio hypothesis) no variable in A_{n+1} at depth s in Θ appears in some A_k $(1 \leq k \leq n)$ at depth s in Θ. Next, we extend v to an interpretation I on $\mathcal{M}_M$ according to clauses (i)-(vi) in Definition 2.6. Then, by Lemma 2.11, we have: (4) $I_{r-k}(A_k) \in K_1$ for each A_k $(1 \leq k \leq n)$; (5) $I_{r-n}(A_{n+1}) \in K_2$.

We can now proceed as follows. By (4), (5), condition (3) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-n}(A_n \overset{M}{\to} A_{n+1}) = a_F$; and by clause (vib) (Definition 2.6), $I_{r-(n-1)}(A_n \to A_{n+1}) = a_F$. Next, by (4), condition (4) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-(n-1)}(A_{n-1} \overset{M}{\to} (A_n \to A_{n+1})) = a_F$, whence by clause (vib) (Definition 2.6), $I_{r-(n-2)}(A_{n-1} \to (A_n \to A_{n+1})) = a_F$. By repeating the argumentation, we get $I_{r-(n-n)}(A_{n-(n-1)} \to (... \to (A_n \to A_{n+1})...)) = a_F$, whence by clause (vic) (Definition 2.6), $I_\omega(A_1 \to (... \to (A_n \to A_{n+1})...)) \notin T$, i.e., $\nvDash_{\mathcal{M}_M} \Theta$, contradicting the hypothesis.

Therefore, if Θ is valid in some wr-ms $\mathcal{M}_M$, then $\Delta(\Theta(A_{n+1}), \Theta(A_k)) \neq \emptyset$ for some A_k $(1 \leq k \leq n)$. ∎

Remark 3.12 (t11-t17 are falsifiable in any wr-ms) *As noted in Remark 2.13, t11-t17 are not ruled out by Theorem 2.12. They are however falsified by Lemma 3.11. Consider the following instances of t11-t17:*

$$\begin{aligned}
&t11'.\ [p \to (p \to q)] \to (p \to q)\\
&t12'.\ [p \to (q \to r)] \to [(p \wedge q) \to r]\\
&T13'.\ (p \to q) \to [p \to (p \to q)]\\
&T14'.\ (p \to q) \to [q \to (p \to q)]\\
&T15'.\ [(p \to q) \to p] \to [(p \to q) \to q]\\
&T16'.\ [(q \to p) \to p] \to [(p \to q) \to q]\\
&T17'.\ [(p \wedge q) \to r] \to [p \to (q \to r)]
\end{aligned}$$

The last variable (from left to right) in each one of t11′-t17′ does not appear at the same depth in the rest of the formula. Therefore, t11-t17 are falsified in any wr-ms by Lemma 3.11.

Nevertheless, we note the following.

Remark 3.13 (Theses t18-22) *Consider the following theses:*

$$\begin{aligned}
&t18.\ [A \to (B \to C)] \to [B \to (A \to C)]\\
&t19.\ [(A \to B) \to (A \to C)] \to [A \to (B \to C)]\\
&t20.\ [(A \to B) \to (A \to C)] \to [B \to (A \to C)]\\
&t21.\ [(A \to B) \to C] \to (B \to C)\\
&t22.\ [(A \to B) \to A] \to (B \to A)
\end{aligned}$$

Lemma 3.11 cannot be used in order to falsify t18-t22 since variables in the last subformula (from left to right) of each one of t18-t22 appear at the same depth somewhere in the rest of the formula. Nevertheless, they are falsifiable by Theorem 3.14, which is proved below.

As we have seen, Lemma 3.11 is provable for any wr-ms and so is the case with Theorem 3.14 which, leaning on Lemma 3.11, records the fundamental fact on α-formulas.

Theorem 3.14 (The main property of α-wffs in wr-ms) *Let $\mathcal{M}_M$ be a wr-ms and (Θ) $A_1 \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ be a $\mathcal{M}_M$-valid α-formula. Then, for each A_k ($i \leq k \leq n+1$) there is some A_i ($i \in \{1, 2, ..., n+1\}$ and $i \neq k$) such that $\Delta(\Theta(A_k), \Theta(A_i)) \neq \emptyset$.*

Proof. Assume the hypothesis of Theorem 3.14. (1) $n = 1$. The proof follows by Theorem 2.12. (2) $n > 1$. (a) $k = 1$. By Theorem 2.12, $\Delta(\Theta(A_1), \Theta(A_i)) \neq \emptyset$ for some A_i ($2 \leq i \leq n+1$). (b) $k = n+1$. By Lemma 3.11, $\Delta(\Theta(A_{n+1}), \Theta(A_i)) \neq \emptyset$ for some A_i ($1 \leq i \leq n$). (c) A_k is a member in the sequence $A_2, ..., A_n$ (i.e., $2 \leq k \leq n$). For reductio, suppose that there is some A_k ($2 \leq k \leq n$) such that

$\Delta(\Theta(A_k), \Theta(A_i)) = \emptyset$ for each A_i $(1 \leq i \leq n+1)$ $(i \neq k)$. We prove that Θ is not $\mathcal{M}_M$-valid, which contradicts the hypothesis. Let $a \in K_1$, $b \in K_2$, $c \in K$ and $deg(\Theta) = r$. Similarly, as in Lemma 3.11, we define an interpretation I on $\mathcal{M}_M$ by leaning on the following v: for each propositional variable p we set: (1) $v_{r-d-k}(p) = a$ for each depth d that p occurs in A_k; (2) $v_{r-d-m}(p) = b$ for each depth d that p occurs in each A_m ($m \in \{1, 2, ..., n+1\}$ and $m \neq k$); (3) $v_j(p) = c$ if $j \geq r$ or $j = r - d - l$ ($l \in \{1, 2, ..., n+1\}$) but p does not occur at depth d in A_l. Then, by Lemma 2.11, we have: (4) $I_{r-k}(A_k) \in K_1$; (5) $I_{r-m}(A_m) \in K_2$ for each A_m ($m \in \{1, 2, ..., n+1\}$ and $m \neq k$).

Now, we can proceed as follows. Suppose $k < n - 1$ (if $k = n$ or $k = n - 1$ the proof is similar). By (5), condition (2) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-n}(A_n \overset{M}{\to} A_{n+1}) \in K_2$; by clause (vib) (Definition 2.6), $I_{r-(n-1)}(A_n \to A_{n+1}) \in K_2$. By repeating the argumentation, we get :$I_{r-(n-(n-k))}(A_{k+1} \to (A_{k+2} \to (... \to (A_n \to A_{n+1})...))) \in K_2$. Now, by (4), $I_{r-(n-(n-k))}(A_k) \in K_1$. So, by condition (3) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-(n-(n-k))}(A_k \overset{M}{\to} (A_{k+1} \to (A_{k+2} \to (... \to (A_n \to A_{n+1})...)))) = a_F$, and then, by clause (vib) (Definition 2.6), $I_{r-(n-(n-(k-1)))}(A_k \to (A_{k+1} \to (A_{k+2} \to (... \to (A_n \to A_{n+1})...)))) = a_F$. On the other hand, by (5), $I_{r-(n-(n-(k-1)))}(A_{k-1}) \in K_2$. So, by condition (4) (Definition 2.1) and clause (v) (Definition 2.6), we get, $I_{r-(n-(n-(k-1)))}(A_{k-1} \overset{M}{\to} (A_k \to (A_{k+1} \to (... \to (A_n \to A_{n+1})...)))) = a_F$, and then, by clause (vib), $I_{r-(n-(n-(k-2)))}(A_{k-1} \to (A_k \to (A_{k+1} \to (... \to (A_n \to A_{n+1})...)))) = a_F$. By repeating the argumentation, we have $I_{r-(n-(n-(k-k)))}(A_{k-(k-1)} \to (A_{k-(k-2)} \to (... \to (A_k \to (A_{k+1} \to (... \to (A_n \to A_{n+1})...)))))) = a_F$. That is, $I_r(A_1 \to (A_2 \to (... \to (A_n \to A_{n+1})...))) = a_F$, whence, by clause (vic) (Definition 2.6), $I_\omega(\Theta) \notin T$, which contradicts the hypothesis, thus ending the proof of case (2) (c).

Therefore, if Θ is valid in some wr-ms $\mathcal{M}_M$, then for each A_k $(1 \leq k \leq n+1)$, there is some A_i ($i \in \{1, 2, ..., n+1\}$ and $i \neq k$) such that $\Delta(\Theta(A_k), \Theta(A_i)) \neq \emptyset$. ■

Remark 3.15 (t18-t22 are falsified in any wr-ms) *As pointed out in Remark 3.13, Lemma 3.11 cannot be employed in falsifying t18-t22. Nevertheless, these theses are falsified by using Theorem 3.14. Consider the following instances of t18-t22 where a variable not appearing at the same depth in the rest of the formula is underlined:*

t18′. $[p \to (q \to r)] \to [\underline{q} \to (p \to r)]$

t19′. $[(p \to q) \to (p \to r)] \to [\underline{p} \to (q \to r)]$

t20′. $[(p \to q) \to (p \to r)] \to [\underline{q} \to (p \to r)]$

t21′. $[(p \to q) \to r] \to (\underline{q} \to r)$

t22′. $[(p \to q) \to p] \to (\underline{q} \to p)$

By Theorem 3.14, it is clear that T18′-T22′ are falsified in any wr-ms. So, t18-t22 are valid in no wr-ms.

3.3 The main properties of β-formulas in wr(1)-ms and wr(1,2)-ms

In what follows, we investigate the main properties of β-formulas in wr(1)-ms and wr(2)-ms. But in order to do this we need the lemmas that follow.

Lemma 3.16 (Interpreting wffs with a_F) *Let $\mathcal{M}_M$ be a wr(1)-ms and A be a weak positive formula (cf. Definition 1.7). Then, for all interpretations I defined on $\mathcal{M}_M$ as indicated below, we prove that $I(B) = a_F$ (in particular, $I(A) = a_F$) for each subformula B of A. Now, I is defined by extending the valuation v according to clauses (i), (iii) and (iv) (Definition 2.6). This valuation is in its turn defined as follows. For each propositional variable p, we set: (1) $v_0(p) = a_F$ if p occurs in A. (2) $v_j(p)$ is arbitrarily assigned if $j \geq 1$ or $j = 0$ but p does not occur in A.*

Proof. We recall that "weak positive formulas" are those in which only the connectives $\wedge$ and $\vee$ appear (cf. Definition 1.7). Consequently, $deg(A) = 0$. Then, the proof of Lemma 3.16 is trivial by using condition (7) in Definition 3.1 (definition of wr(1)-ms). ■

Lemma 3.17 (Interpreting wffs with a_T) *Let $\mathcal{M}_M$ be a wr(2)-ms and A be a positive formula (cf. Definition 1.7) of degree r. Then, we define an interpretation I on $\mathcal{M}_M$ based on the following valuation v. For each propositional variable p, we set: (1) $v_{r-d}(p) = a_T$ for each depth d that p occurs in A. (2) $v_j(p)$ is arbitrarily assigned if $j > r$ or $j = r - d$ but p does not occur at depth d in A. Then, we have $I_{r-d}(B) = a_T$ (in particular, $I_r(A) = a_T$) for each subformula B of A and depth d that B occurs in A.*

Proof. We recall that a "positive formula" is a wff in which $\neg$ does not appear (cf. Definition 1.7). Then, the proof is similar to those of Lemma 2.11 and Lemma 3.16 (use condition (9) in Definition 3.3). ■

Once we have lemmas 3.16 and 3.17 at our disposal we can prove theorems 3.18 and 3.19.

Theorem 3.18 (The main property of β-wffs in wr(1)-ms) *Let $\mathcal{M}_M$ be a wr(1)-ms and (Θ) $[(...((A_{n+1} \to A_n) \to A_{n-1}) \to ...) \to A_2] \to A_1$ be a $\mathcal{M}_M$-valid β-formula. Suppose, further, that A_k ($1 \leq k \leq n+1$) is a weak positive formula (cf. Definition 1.7) and $k = 1$ or k is an odd number. Then, there is some A_i ($i \in \{1, 2, ..., n+1\}$ and $i \neq k$) such that $\Delta(\Theta(A_k), \Theta(A_i)) \neq \emptyset$.*

Proof. Assume the hypothesis of Theorem 3.18. (1) $n = 1$. The proof follows by Theorem 2.12. (2) $n > 1$. (a) $k = 1$. By Theorem 2.12, there is some A_i ($2 \leq i \leq n+1$) such that $\Delta(\Theta(A_1), \Theta(A_i)) \neq \emptyset$. (b) k is an odd number. Let A_k be a weak positive formula in the sequence $A_2, A_3, ..., A_n, A_{n+1}$, being k an odd number. Suppose, for reductio, that for each A_i ($i \in \{1, 2, ..., n, n+1\}$ and $i \neq k$), $\Delta(\Theta(A_k), \Theta(A_i)) = \emptyset$. We prove that Θ is not $\mathcal{M}_M$-valid, which contradicts the hypothesis.

Let $b \in K_2$, $c \in K$ and $deg(\Theta) = r$ (notice that $deg(A_k) = 0$ since A_k is a weak positive formula). As in Theorem 3.14, we define an interpretation I on $\mathcal{M}_M$ leaning on the valuation v, which is defined as follows. For each propositional variable p, we set: (1) $v_{r-k}(p) = a_F$ for each variable p in A_k (notice that for each variable p in A_k, $d[p, A_k] = 0$ since $deg(A_k) = 0$); (2) $v_{r-d-m}(p) = b$ for each depth d that p appears in each A_m ($m \in \{1, 2, ..., n+1\}$ and $m \neq k$); (3) $v_j(c) = c$ if $j \geq r$ or $j = r - d - l$ ($l \in \{1, 2, ..., n+1\}$) but p does not occur at depth d in A_l. Then, by Lemma 2.11 and Lemma 3.16, we have: (4) $I_{r-k}(A_k) = a_F$; (5) $I_{r-m}(A_m) \in K_2$ for each A_m ($m \in \{1, 2, ..., n+1\}$ and $k \neq m$).

Now, we can proceed as follows. As in Theorem 3.14, we can suppose that $k < n-1$ given that if $k = n$ or $k = n-1$, then the proof is similar to the one we are going to display.

By (5), condition (2) (Definition 2.1) and clause (v) (Definition 2.6), I_{r-n} $(A_{n+1} \overset{M}{\to} A_n) \in K_2$; by clause (vib) (Definition 2.6), $I_{r-(n-1)}(A_{n+1} \to A_n) \in K_2$. By repeating the argumentation, we get : $I_{r-(n-(n-k))}([(...(A_{n+1} \to A_n) \to ...) \to A_{k+2}] \to A_{k+1}) \in K_2$. Now, by (4), $I_{r-(n-(n-k))}(A_k) = a_F$. So, by condition (4) (Definition 2.1) and clause (v) (Definition 2.6), $I_{r-(n-(n-k))}([(...(A_{n+1} \to A_n) \to ...) \to A_{k+1}] \overset{M}{\to} A_k) = a_F$. Then, by clause (vib) (Definition 2.6), $I_{r-(n-(n-(k-1)))}([(...(A_{n+1} \to A_n) \to ...) \to A_{k+1}] \to A_k) = a_F$. On the other hand, by (5), $I_{r-(n-(n-(k-1)))}(A_{k-1}) \in K_2$. So, by condition (8) (Definition 3.1), $I_{r-(n-(n-(k-1)))}([(...(A_{n+1} \to A_n) \to ...) \to A_k] \overset{M}{\to} A_{k-1}) = a_T$, and then, by clause (vib) (Definition 2.6), $I_{r-(n-(n-(k-2)))}([(...(A_{n+1} \to A_n) \to ...) \to A_k] \to A_{k-1}) = a_T$. Next, again by (5), $I_{r-(n-(n-(k-2)))}(A_{k-2}) \in K_2$; and by condition (6) (Definition 3.1), $I_{r-(n-(n-(k-2)))}([(...(A_{n+1} \to A_n) \to ...) \to A_{k-1}] \overset{M}{\to} A_{k-2}) = a_F$, whence, by clause (vib) (Definition 2.6), $I_{r-(n-(n-(k-3)))}$ $([(...(A_{n+1} \to A_n) \to ...) \to A_{k-1}] \to A_{k-2}) = a_F$. By repeating the argumentation, $I_{r-(n-(n-(k-(k-1))))}([(...(A_{n+1} \to A_n) \to ...) \to A_{k-(k-2)}] \overset{M}{\to} A_{k-(k-1)}) = a_F$, and, finally, (clause (vib), Definition 2.6), $I_{r-(n-(n-(k-k)))}([(...$ $(A_{n+1} \to A_n) \to ...) \to A_{k-(k-2)}] \to A_{k-(k-1)}) = a_F$, that is, $I_r([(...(A_{n+1} \to A_n) \to ...) \to A_2] \to A_1) = a_F$ whence $I_\omega(\Theta) \notin T$ (clause (vic) —Definition 2.6) contradicting the hypothesis, thus ending the proof of case (2) (b).

Therefore, if $\mathcal{M}_M$ is a wr(1)-ms and Θ is a $\mathcal{M}_M$-valid β-formula and A_k ($1 \leq k \leq n+1$) is a weak positive formula ($k = 1$ or k is an odd number), then there is some variable common to A_k and some A_i ($i \in \{1, 2, ..., n+1\}$ and $i \neq k$) at the same depth in Θ. ∎

As we have seen, Theorem 3.18 is proved for wr(1)-ms provided A_k ($1 \leq k \leq n$) in Θ is a weak positive formula where $k = 1$ or k is an odd number. However, if k is even, then wr(1)-ms are not sufficient and wr(1, 2)-ms are required. But, on the other hand, weak positive formulas can be strengthened to positive ones.

Theorem 3.19 (The main property of β-wffs in wr(1, 2)-ms) *Let $\mathcal{M}_M$ be a wr(1, 2)-ms and* (Θ) $[(...((A_{n+1} \to A_n) \to A_{n-1}) \to ...) \to A_2] \to A_1$ *be a $\mathcal{M}_M$-valid β-formula. Further, suppose that A_k $(1 \leq k \leq n+1)$ is a posi-*

tive formula (cf. Definition 1.7) and $k = 1$ or k is an even number. Then, there is some A_i ($i \in \{1, 2, ..., n+1\}$ and $i \neq k$) such that $\Delta(\Theta(A_k), \Theta(A_i)) \neq \Theta$.

Proof. It is similar to that of Theorem 3.18. Thus, it will suffice to define the valuation v on which the interpretation I on $\mathcal{M}_M$ is built in order to proceed by reductio ad absurdum as above. Now, let $b \in K_2$, $c \in K$ and $deg(\Theta) = r$. For each propositional variable p, we set: (1) $v_{r-d-k}(p) = a_T$ for each depth d that p appears in A_k; (2) $v_{r-d-m}(p) = b$ for each depth d that p appears in each $A_m (m \in \{1, 2, ..., n+1\}$ and $m \neq k)$; (3) $v_j(p) = c$ if $j \geq r$ or $j = r - d - l$ ($l \in \{1, 2, ..., n+1\}$) but p does not occur at depth d in A_l. Then, the details completing the proof are left to the reader. ■

Remark 3.20 (t23 and t24 are falsified in wr(1)-ms) *Consider the following wffs:*

t23. $(q \to p) \to (p \to p)$
t24. $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
t25. $(p \to q) \to (p \to p)$
t26. $[(p \to q) \to r] \to [(p \to q) \to (p \to r)]$
t27. $\{[(p \to p) \to p] \to p\} \to [(p \to p) \to (p \to p)]$

t23-t27 cannot be falsified by using Theorem 2.12, Lemma 3.11 or Theorem 3.14. But t23 and t24 are falsified by Theorem 3.18 and t25, t26 and t27 are falsified by Theorem 3.19 (notice however that t25, t26 and t27 cannot be falsified by Theorem 3.18).

Finally, we note that in order to invalidate the routes in Theorem 1.6, it suffices Theorem 3.23 (1), an instance of Theorem 3.19 proved below, and valid in any wr(2)-ms.

3.4 Other properties of wr(1)-ms and wr(2)-ms. Wr-model structures of type 3

In what follows we prove a couple of properties that will be needed in the following section. One is predicable of wr(1)-ms and the other of wr(2)-ms. Then, we shall define wr(3)-ms and prove some general schemes not valid in w(3)-ms. The proofs are similar to those developed above and we shall limit ourselves to a sketch of the required interpretation on the wr-ms of the case considered.

Theorem 3.21 (Two more schemes not valid wr(1)-ms) *Let $\mathcal{M}_M$ be a wr(1)-ms and (Θa) $(A_5 \to A_4) \to [(A_3 \to A_2) \to A_1]$, ($\Theta b$) $(A_5 \to A_4) \to [(A_2 \to A_3) \to A_1]$ be wffs where $deg(A_2) \geq 1$, $deg(A_4) \geq 1$ and $deg(A_1) = deg(A_3) = deg(A_5) = 0$. Then, (1) $\nvDash_{\mathcal{M}_M} \Theta a$ and (2) $\nvDash_{\mathcal{M}_M} \Theta b$.*

Proof. Assume the hypothesis of Theorem 3.21. We prove case (1) (the proof of case (2) is similar). Let r represent any propositional variable in this theorem and in the rest of the theorems of the present section. Then, $d[r, \Theta a(A_1)] = d[r, \Theta a(A_5)] = 2$, $d[r, \Theta a(A_2)] \geq 4$, $d[r, \Theta a(A_3)] = 3$ and $d[r, \Theta a(A_4)] \geq 3$. Next, we define an interpretation I on $\mathcal{M}_M$ assigning a_F to r at each depth d that r appears in A1 and A5; and assigning b ($b \in K_2$) to r at each depth d that r occurs in A_2, A_3 and A_4. Then, it is easy to show that $I_\omega(\Theta a) \notin T$, as was to be proved. ■

Remark 3.22 (t28 and t29 are falsified in wr(1)-ms) *The wffs*

$$t28.\ [p \to (p \to p)] \to [[(p \to p) \to p] \to p]$$
$$t29.\ [p \to (p \to p)] \to [[p \to (p \to p)] \to p]$$

are instances of Θa *and* Θb *(Theorem 3.21), respectively (notice that t28 and t29 cannot be falsified by using the procedure in the proof of Theorem 3.18).*

Theorem 3.23 (Two more schemes not valid in wr(2)-ms) *Let* $\mathcal{M}_M$ *be a wr(2)-ms and* (Θa) $(B \to C) \to D$, (Θb) $(A_5 \to A_4) \to [(A_3 \to A_2) \to A_1]$ *be wffs where* C *and* A_2 *are positive formulas,* $\Delta(\Theta a(C), \Theta a(B)) = \Delta(\Theta a(C), \Theta a(D)) = \emptyset$, $deg(A_2) \geq 1$ *and* $deg(A_5) = 1$, $deg(A_1) = deg(A_3) = deg(A_4) = 0$. *Then, (1)* $\nvDash_{\mathcal{M}_M} \Theta a$ *and (2)* $\nvDash_{\mathcal{M}_M} \Theta b$.

Proof. Case 1. Define an interpretation I on $\mathcal{M}_M$ assigning a_T to r at each depth d that r appears in C; and assigning b ($b \in K_2$) to r at each depth d that r occurs in B and D. Then, clearly, $I_\omega(\Theta a) \notin T$, as was to be proved (notice that if $\mathcal{M}_M$ is a wr(1, 2)-ms, then case 1 is immediate by Theorem 3.19). Case 2. We remark that $d[r, \Theta b(A_1)] = d[r, \Theta b(A_4)] = 2$, $d[r, \Theta b(A_3)] = d[r, \Theta b(A_5)] = 3$ and $d[r, \Theta b(A_2)] \geq 4$. Thus, we can define an interpretation I on $\mathcal{M}_M$ assigning a_T to r at each depth d that r appears in A_2; and assigning $b(b \in K_2)$ to r at each depth d that r appears in A_1, A_3, A_4 and A_5. Then, $I_\omega(\Theta b) \notin T$, as it was required. ■

Remark 3.24 (t30 and t31 are falsified in wr(2)-ms) *The wffs*

$$t30.\ [p \to (p \to p)] \to (p \to p)$$
$$t31.\ [(p \to p) \to p)] \to [[p \to (p \to p)] \to p]$$

are instances of Θa *and* Θb *(Theorem 3.23), respectively (notice that t25, t26 and t27 are instances of* Θa*, cf. Remark 3.20).*

Next, wr(3)-ms are defined and a couple of properties of w(3)-ms are proved.

Definition 3.25 (Weak relevant matrices. Type 3) *Let* M *be a wr-matrix (cf. Definition 2.1) with* e, $e' \in K$. *Furthermore,* M *fulfills the following conditions (in addition to (1)-(4); cf. Definition 2.1): (11)* $f_\wedge(e, e) = f_\vee(e, e) = f_\to(e, e) = e$; *(12)* $f_\wedge(e', e') = f_\vee(e', e') = e'$; *(13)* $f_\to(e', e') = e$; *(14)* $f_\to(e, e') = e'$; *(15)* $f_\to(e', e) = a_F$; *(16)* $f_\to(e, a_F) = a_F$. *Then,* M *is a weak relevant matrix of type 3 (wr(3)-matrix, for short).*

Example 3.26 (Some wr(3)-matrices) *The following matrices in Appendix 2 are wr(3) matrices:* $M1$, $M5$, $M6$, $M7$ *and* $M8$.

Remark 3.27 (wr(3)-ms, interpretations, validity, wr(1, 2, 3)-ms) *Wr(3)-model structures are defined for wr(3)-matrices in a similar way to which the rest of the wr-ms were defined (cf. Definition 3.7). Next, wr(1, 2, 3)-matrices and wr(1, 2, 3)-ms are defined in a similar way to which wr(1, 2)-matrices and wr(1, 2)-ms were defined (cf. Definition 3.5; Definition 3.7). Then, valuations, interpretations, validity and the notion "a logic verified by a wr(3)-ms" are understood as in Remark 3.9. Finally, particular wr(3)-ms (wr(1, 3)-ms, wr(2, 3)-ms, wr(1, 2, 3)-ms) are built upon wr(3)-matrices (wr(1, 3)-matrices, wr(2, 3)-matrices, wr(1, 2, 3)-matrices) in Appendix 2 as above (cf. Example 2.9).*

We end the section with the following theorem and a couple of remarks.

Theorem 3.28 (A couple of schemes not valid in wr(3)-ms) *Let* $\mathcal{M}_M$ *be a wr(3)-ms and* (Θa) $(A_4 \to A_3) \to (A_2 \to A_1)$; (Θb) $(A_5 \to A_4) \to [(A_3 \to A_2) \to A_1]$ *be wffs such that (a)* Θa: A_4 *is a positive wff and* $deg(A_4) \geq 1$; *and* $deg(A_1) = deg(A_2) = deg(A_3) = 0$. *(b)* Θb: A_3 *and* A_5 *are positive formulas and* $deg(A_3) \geq 1$, $deg(A_5) = 1$; *and* $deg(A_1) = deg(A_2) = deg(A_4) = 0$. *Then, (1)* $\nvDash_{\mathcal{M}_M} \Theta a$ *and (2)* $\nvDash_{\mathcal{M}_M} \Theta b$.

Proof. It is similar to those of Theorem 3.21 and Theorem 3.23. Case 1. Define an interpretation I on $\mathcal{M}_M$ assigning e to r at each depth d that r occurs in A_4; and assigning e' to r at each depth d that r occurs in A_1, A_2 and A_3. Then, $\nvDash_{\mathcal{M}_M} \Theta a$. Case 2. As $deg(A_5) = 1$, A_5 is of the form $B \to C$ with $deg(B) = deg(C) = 0$. Now, define an interpretation I on $\mathcal{M}_M$ assigning e' to r at each depth d that r appears in B, C and A_2, and assigning e to r at each depth d that r appears in A_1, A_3 and A_4. Then, it is easy to show that $\nvDash_{\mathcal{M}_M} \Theta b$. ■

Remark 3.29 (t32 and t33 are falsified in wr(3)-ms) *The wffs*

$$t32.\ [(p \to p) \to p)] \to (p \to p)$$
$$t33.\ [(p \to p) \to p)] \to [[(p \to p) \to p] \to p]$$

are instances of Θa *and* Θb *(Theorem 3.28), respectively.*

Remark 3.30 (On t1-t33) *The theses t1-t33 have been remarked in this and the preceding section. We note that t7, t11 and t28-t33 belong to some of the classes defined by Rogerson and Restall (cf. Definition 1.5), but the rest of them do not.*

4 Blocking the routes to triviality

In this section, we show how to block the routes to triviality defined in Rogerson and Restall Theorem (Theorem 1.6). We begin by defining and invaliding two important subclasses of α_ψ-formulas (cf. Definition 1.5).

4.1 Invaliding γ-formulas and δ-formulas

Firstly, we set:

Definition 4.1 (γ-formulas, δ-formulas) *Let $\psi(p,q)$ be as in Definition 1.3. That is, $\psi(p,q)$ is an arbitrary but fixed formula containing no variables other than p and q. And let Θ be an α_ψ-formula (cf. Definition 1.5). In other words, Θ is a formula of the form (Θ) $A_1 \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ where, for each i $(1 \le i \le n)$, $A_i \in V_\psi$ (i.e, A_i is either of the form $p \to \psi(p,q)$ or $\psi(p,q) \to p$) and $A_{n+1} \in U_\psi$ (i.e., A_{n+1} is p, q or $\psi(p,q)$). Then, Θ is a γ-formula if A_1 is of the form $p \to \psi(p,q)$; and Θ is a δ-formula if A_1 is of the form $\psi(p,q) \to p$.*

Remark 4.2 (Notational convention) *We shall use r, U and A_k $(1 \le k \le n)$, respectively, when generally referring to the following wffs appearing in the α_ψ-formulas in the subsequent proofs: (1) any propositional variable; (2) any member of U_ψ; (3) the $k-th$ element A_k (be it of the form $p \to \psi(p,q)$ or $\psi(p,q) \to p$) in the α_ψ-formula. On the other hand, before proceeding into the proofs, it may now be convenient to consult Remark 3.10.*

In what follows, we show how to invalidate general classes of γ-formulas and δ-formulas in the different types of wr-ms.

Proposition 4.3 (α_ψ-formulas where $deg(\psi(p,q)) = 0$) *Let $\mathcal{M}_M$ be a wr-ms and Θ be an α_ψ-formula where $deg(\psi(p,q)) = 0$. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.3. We prove that Θ is not $\mathcal{M}_M$-valid if it is a γ-formula (if Θ is a δ-formula, the proof is similar). (1) $n = 1$. Then, Θ is of the form $(p \to \psi(p,q)) \to U$ and we have $d[r, \Theta(A_1)] = 2$ and $d[r, \Theta(U)] = 1$. So, $\Delta(\Theta(A_1), \Theta(U)) = \emptyset$, and, thus, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 2.12. (2) $n = 2$. Suppose Θ is of the form $(p \to \psi(p,q)) \to ((p \to \psi(p,q)) \to U)$ (if Θ is of the form $(p \to \psi(p,q)) \to ((\psi(p,q) \to p) \to U)$, the proof is similar). Then, we have $d[r, \Theta(A_1)] = d[r, \Theta(U)] = 2$ but $d[r, \Theta(A_2)] = 3$. So, $\Delta(\Theta(A_2), \Theta(A_1)) = \emptyset$ and $\Delta(\Theta(A_2), \Theta(U)) = \emptyset$, and thus, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 3.14. (3) $n \ge 3$. Then, for any A_k $(2 \le k \le n+1)$, $d[r, \Theta(A_k)] \ge 3$. So, Θ is of the form $A_1 \to B$ where $\Delta(\Theta(A_1), \Theta(B)) = \emptyset$. Consequently, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 2.12. ■

Proposition 4.4 (α_ψ-wffs with $n = 1$, $deg(\psi(p,q)) \ge 1$ and $deg(U) = 0$) *Let $\mathcal{M}_M$ be a wr-ms and Θ be an α_ψ-formula where $n = 1$, $deg(\psi(p,q)) \ge 1$ and $deg(U) = 0$. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.4. Suppose that Θ is a γ-formula (if Θ is a δ-formula, the proof is similar). Firstly, notice that as $deg(\psi(p,q)) \ge 1$ and $deg(U) = 0$, then U is either p or else q. Secondly, Θ is of the form $(p \to \psi(p,q)) \to U$ where $d[\overset{1}{p}, \Theta(A_1)] = 2$, $d[r, \Theta(A_1(\psi(p,q))] \ge 3$ and $d[r, \Theta(U)] = 1$ (cf. Remark 1.11 on notation). So, Θ is of the form $B \to C$ with $\Delta(B, C) = \emptyset$. Thus, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 2.12. ■

The simple proof of Proposition 4.3 displayed above leans on the following fact. Let Θ be an α_ψ-formula, $deg(\psi(p,q)) = 0$ and $k > m$ $(k, m \in \{1, 2, ..., n\})$. Then, $d[r, \Theta(A_k)] > d[r, \Theta(A_m)]$ (cf. Remark 3.10). However, this is not the case if $deg(\psi(p,q)) \geq 1$. Let us illustrate the situation with a simple example. Consider the following member Θ of $Z\psi$ (cf. Definition 1.5) where $n = 2$ and $deg(\psi(p,q)) = 1$ and let $\psi(p,q)$ be, say, $p \to q$: (Θ) $[p \to (p \to q)] \to [[p \to (p \to q)] \to p]$. We have $d[\overset{1}{p}, \Theta] = d[\overset{0}{p}, \Theta] = 2$ and $d[\overset{2}{p}, \Theta(A_1)] = d[\overset{1}{p}, \Theta(A_2)] = 3$. So, the tools used in the proof of Proposition 4.3 (i.e., theorems 2.12 and 3.14) do not work for invalidating Θ. Similar considerations are in order in the case of Proposition 4.4 if $n \geq 2$. In other words, properties of wr-ms are insufficient for invalidating α_ψ-wffs of more complex structure than that in propositions 4.3 and 4.4 and, consequently, from now on, we shall need to rely on properties of wr(1)-ms, wr(2)-ms and wr(3)-ms, as the case may be.

Proposition 4.5 **(α_ψ-wffs with $n \geq 3$ and $deg(\psi(p,q)) \geq 1$)** *Let $\mathcal{M}_M$ be a wr(1)-ms, $\mathcal{M}'_M$ be a wr(2)-ms and Θ an α_ψ-formula where $n \geq 3$ and $deg(\psi(p,q)) \geq 1$. Then, (1) if Θ is a γ-formula, then $\nvDash_{\mathcal{M}_M} \Theta$; (2) if Θ is a δ-formula, then $\nvDash_{\mathcal{M}'_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.5. (1) Θ is a γ-formula. Then, Θ is of the form $(p \to \psi(p,q)) \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ (On the other hand, recall that $A_k \in V_\psi$ $(2 \leq k \leq n)$). Now, we have $d[\overset{1}{p}, \Theta(A_1)] = 2$, $d[r, \Theta(A_1(\psi(p,q)))] \geq 3$, $d[r, \Theta(A_{n+1})] \geq 3$ and for each A_k $(2 \leq k \leq n)$, $d[r, \Theta(A_k)] \geq 3$. So, Θ can be read as a formula of the form $(B \to C) \to D$ where B (i.e., $\overset{1}{p}(A_1)$) does not share variables at the same depth in Θ with C (i.e., $\psi(p,q)(A1)$) and D (i.e., $A_2 \to (... \to (A_n \to A_{n+1})...)$). Consequently, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 3.18. (2) Θ is a δ-formula. Then, Θ is of the form $(\psi(p,q) \to p) \to [A_2 \to (... \to (A_n \to A_{n+1})...)]$ where $A_k \in V_\psi$ $(2 \leq k \leq n)$. Now, as in case (1), Θ can be read as a formula of the form $(B \to C) \to D$ where C (i.e., $\overset{0}{p}(A_1)$) does not share variables at the same depth with B (i.e., $\psi(p,q)(A_1)$) and D (i.e., $A_2 \to (... \to (A_n \to A_{n+1})...)$). Then, $\nvDash_{\mathcal{M}'_M} \Theta$ by Theorem 3.23(1). ■

Proposition 4.6 **(α_ψ-wffs with $n = 2$, $deg(\psi(p,q)) \geq 1$ and $U \neq p$)** *Let $\mathcal{M}_M$ be a wr(1)-ms, $\mathcal{M}'_M$ be a wr(2)-ms and Θ an α_ψ-formula where $n = 2$ and $deg(\psi(p,q)) \geq 1$ and $U \neq p$. Then, (1) if Θ is a γ-formula, then $\nvDash_{\mathcal{M}_M} \Theta$; (2) if Θ is a δ-formula, then $\nvDash_{\mathcal{M}'_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.6. (1) Θ is a γ-formula. Suppose that Θ is a γ-formula where A_2 is of the form $p \to \psi(p,q)$ (if A_2 is of the form $\psi(p,q) \to p$, the proof is similar). Thus, Θ is of the form $(p \to \psi(p,q)) \to ((p \to \psi(p,q)) \to U)$ with $deg(\psi(p,q)) \geq 1$ and $U \neq p$. Then, $d[\overset{1}{p}, \Theta(A_1)] = 2$, $d[r, \Theta(A_1(\psi(p,q)))] \geq 3$. Now, if U is q, then $d[q, \Theta(U)] = 2$; and if U is $\psi(p,q)$, then $d[r, \Theta(U)] \geq 3$. So, as in Proposition 4.5, Θ can be read as a formula of the form $(B \to C) \to D$ where $\Delta(\Theta(B), \Theta(C)) = \Delta(\Theta(B), \Theta(D)) = \emptyset$.

Consequently, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 3.18. (2) Θ is a δ-formula. In a similar way, it can be shown that Θ can be read as a formula of the form $(B \to C) \to D$ where $\Delta(\Theta(C), \Theta(B)) = \Delta(\Theta(C), \Theta(D)) = \emptyset$. then $\nvDash_{\mathcal{M}'_M} \Theta$ by Theorem 3.23(1). ■

4.2 Blocking the routes to triviality

In what follows, we show how to invalidate members in the classes X_ψ, Y_ψ and Z_ψ (cf. Definition 1.5). We begin by proving that any member in X_ψ is falsified in no matter which wr(1, 2)-ms.

Theorem 4.7 (X_ψ-wffs are falsified in any wr(1, 2)-ms) *Let $\mathcal{M}_M$ be a wr (1, 2)-ms and $\Theta \in X_\psi$. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Theorem 4.7. Then, Θ is of the form (Θ) $A_1 \to [A_2 \to (... \to A_n \to q)...)]$ (cf. Definition 1.5) where $A_k \in V_\psi$ $(1 \leq k \leq n)$. Now, it can be shown that Θ is not $\mathcal{M}_M$-valid as follows. (1) $deg(\psi(p,q)) = 0$. By Proposition 4.3. (2) $deg(\psi(p,q)) \geq 1$. (a) $n = 1$. By Proposition 4.4. (b) $n = 2$. By Proposition 4.6. (c) $n \geq 3$. By Proposition 4.5. ■

Next, we specify some very general circumstances under which members in Y_ψ and in Z_ψ are falsified in wr(1, 2)-ms.

Theorem 4.8 (Y_ψ-wffs where $n \neq 1$ or $deg(\psi(p,q)) = 0$) *Let $\mathcal{M}_M$ be a wr(1, 2)-ms and $\Theta \in Y_\psi$. If $n \neq 1$ or $deg(\psi(p,q)) = 0$, then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Theorem 4.8. Now, Θ is a formula of the form (Θ) $A_1 \to [A_2 \to (... \to (A_n \to \psi(p,q))...)]$ (cf. Definition 1.5) where $A_k \in V_\psi$ $(1 \leq k \leq n)$. Then Θ can be falsified in $\mathcal{M}_M$ as follows. (1) $deg(\psi(p,q)) = 0$. By Proposition 4.3. (2) $n \neq 1$. We can suppose $deg(\psi(p,q)) \geq 1$ (if $deg(\psi(p,q)) = 0$, then the proof follows by case (1)). (a) $n = 2$. By Proposition 4.6. (b) $n \geq 3$. By Proposition 4.5. ■

Theorem 4.9 (Z_ψ-wffs where $n \neq 2$ or $deg(\psi(p,q)) = 0$) *Let $\mathcal{M}_M$ be a wr(1, 2)-ms and $\Theta \in Z_\psi$. If $n \neq 2$ or $deg(\psi(p,q)) = 0$, then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. We recall that Θ is of the form (cf. Definition 1.5) (Θ) $A_1 \to [A_2 \to (... \to (A_n \to p)...)]$. Then, the proof is similar to that of Theorem 4.8 by using now Proposition 4.3, Proposition 4.4 and Proposition 4.5. ■

Now, we proceed into the falsification of members in W_ψ (cf. Definition 1.5).

Proposition 4.10 (W_ψ-wffs of the form $p \to (\psi(p,q) \to q)$) *Let $\mathcal{M}_M$ be a wr-ms and Θ be a member of W_ψ of the form (Θ) $p \to (\psi(p,q) \to q)$. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Immediate by Theorem 2.12: $\Delta(\Theta(p), \Theta(\psi(p,q) \to q) = \emptyset$. ■

Proposition 4.11 (W_ψ-wffs of the form $\psi(p,q) \to (p \to q)$ and $deg(\psi(p,q)) = 0$) *Let $\mathcal{M}_M$ be a wr-ms and Θ be a member of W_ψ of the form (Θ) $\psi(p,q) \to (p \to q)$. If $deg(\psi(p,q)) = 0$, then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.11. Then, $d[r, \Theta(\psi(p, q))] = 1$ but $d[r, \Theta(p \to q)] = 2$. So, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 2.12. ■

Proposition 4.12 (W_ψ-wffs of the form $\psi(p, q) \to (p \to q)$ and $deg(\psi(p, q)) > 1$) *Let $\mathcal{M}_M$ be a wr(2, 3)-ms and Θ be a member of W_ψ of the form (Θ) $\psi(p, q) \to (p \to q)$. If $\psi(p, q)$ is a positive formula (i.e., a wff in which negation does not appear —Definition 1.7) such that $deg(\psi(p, q)) > 1$, then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of the Proposition 4.12 and notice that $d[r, \Theta(p \to q)] = 2$. Then, Θ is falsified in $\mathcal{M}_M$ as follows. (1) $\psi(p, q)$ is of the form $(B \to C) \to (D \to E)$. Then, $d[r, \Theta(\psi(p, q))] \geq 3$ and the proof follows by Theorem 2.12. (2) $\psi(p, q)$ is of the form $(B \to C)$ where $deg(B) = 0$. Then $deg(C) \geq 1$ and thus, $d[r, \Theta(C)] \geq 3$. So, Θ is falsified by using Theorem 3.23(1). (3) $\psi(p, q)$ is of the form $B \to C$ where $deg(C) = 0$. Then, $deg(B) \geq 1$, and thus, $d[r, \Theta(B)] \geq 3$. So, Θ is falsified by using Theorem 3.28(1). ■

Leaning on propositions 4.10, 4.11 and 4.12, we have:

Theorem 4.13 (Cases when W_ψ are falsified in wr(2, 3)-ms) *Let $\mathcal{M}_M$ be a wr(2, 3)-ms and $\Theta \in W_\psi$. Then, $\nvDash_{\mathcal{M}_M} \Theta$ if (1) Θ is of the form $p \to (\psi(p, q) \to q)$; (2) Θ is of the form $\psi(p, q) \to (p \to q)$ and $\psi(p, q)$ is a positive formula and $deg(\psi(p, q)) \neq 1$.*

Proof. Assume the hypothesis of Theorem 4.13. Case (1). The proof follows by Proposition 4.10. Case (2). The proof follows by Proposition 4.11 and Proposition 4.12. ■

Now, let us note the following remarks.

Remark 4.14 (General schemes not falsified) *The general schemes that follow have not been falsified in any of the wr-ms defined: (1) Y_ψ-wffs where $n = 1$ and $deg(\psi(p, q)) \geq 1$; (2) Z_ψ-wffs where $n = 2$ and $deg(\psi(p, q)) \geq 1$; (3) W_ψ-wffs of the form $\psi(p, q) \to (p \to q)$ where $deg(\psi(p, q)) = 1$.*

Remark 4.15 (Routes to triviality still open) *So, the following routes to triviality in Rogerson and Restall Theorem are still open (cf. Theorem 3 in [17]; cf. Theorem 1.6 above). Let B, C, D be provable in a logic S (cf. Definition 1.2). Then, NC trivializes S if one of the following is the case where $deg(\psi(p, q)) = 1$, D is of the form $\psi(p, q) \to (p \to q)$ and $n = 1$ in B and $n = 2$ in C: (1) $B \in Y_\psi$ and $D \in W_\psi$; (2) $C \in Z_\psi$ and $D \in W_\psi$.*

In the rest of this section, we shall block the routes (1) and (2) in Remark 4.15 by showing:

1. Let $B \in Y_\psi$ where $n = 1$. If $\psi(p, q)$ is a positive formula and $deg(\psi(p, q)) = 1$, then B is falsified in any wr(2, 3)-ms.
2. Let $B \in Z_\psi$ where $n = 2$. If $\psi(p, q)$ is a positive formula and $deg(\psi(p, q)) = 1$, then B is falsified in any wr(1, 2, 3)-ms.

Proposition 4.16 (Blocking route (a)) *Let $\mathcal{M}_M$ be a wr(2, 3)-ms and $\Theta \in Y_\psi$ where $n = 1$ and $\psi(p, q)$ is a positive formula such that deg($\psi(p, q)$) = 1. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.16. Then, Θ is either (Θa) $(p \to \psi(p, q)) \to \psi(p, q)$ or (Θb) $(\psi(p, q) \to p) \to \psi(p, q)$. Now, $d[r, \Theta a(\overset{1}{\psi}(p, q))] = d[r, \Theta b(\overset{1}{\psi}(p, q))] = 3$; $d[r, \Theta a(\overset{2}{\psi}(p, q))] = d[r, \Theta b(\overset{2}{\psi}(p, q))] = 2$, and, finally, $d[\overset{1}{p}, \Theta a] = d[\overset{0}{p}, \Theta b(A_1)] = 2$. Next, we show that Θa and Θb are falsified in $\mathcal{M}_M$. (1) Θ is Θa. Then, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 3.23 (1). (2) Θ is Θb. As $deg(\psi(p, q)) = 1$, then $\psi(p, q)$ is of the form $B \to C$ and so, Θb is of the form $[(B \to C) \to p] \to (B \to C)$. Then, $\nvDash_{\mathcal{M}_M} \Theta$ by Theorem 3.28 (1). ■

Proposition 4.17 (Blocking route (b)) *Let $\mathcal{M}_M$ be a wr(1, 2, 3)-ms and $\Theta \in Z_\psi$ where $n = 2$ and $\psi(p, q)$ is a positive formula such that deg($\psi(p, q)$) = 1. Then, $\nvDash_{\mathcal{M}_M} \Theta$.*

Proof. Assume the hypothesis of Proposition 4.17. Then, Θ is of one of the following forms: (Θa) $(p \to \psi(p, q)) \to ((p \to \psi(p, q)) \to p)$; (Θb) $(p \to \psi(p, q)) \to ((\psi(p, q) \to p) \to p)$; (Θc) $(\psi(p, q) \to p) \to ((p \to \psi(p, q)) \to p)$; (Θd) $(\psi(p, q) \to p) \to ((\psi(p, q) \to p) \to p)$. Now, bearing in mind that $deg(p) = 0$ and $deg(\psi(p, q)) = 1$, Θ is shown to be not $\mathcal{M}_M$-valid as follows. Case (1). Θ is Θa or Θb. Then, Θ is falsified by Theorem 3.21. Case (2). Θ is Θc. then, Θ is falsified by Theorem 3.23 (2). Case 3. Θ is Θd. Then, Θ is falsified by Theorem 3.28 (2). ■

Finally, we record the results obtained in the following theorem.

Theorem 4.18 (The routes to triviality, blocked) *(1) Let X_ψ be defined from any $\psi(p, q)$-wff (cf. Definition 1.5). Then, if $A \in X_\psi$, $\nvDash_{\mathcal{M}_M} A$ for any wr(1, 2)-ms $\mathcal{M}_M$. (2) Let Y_ψ, Z_ψ and W_ψ be defined from a positive $\psi(p, q)$-wff (cf. Definition 1.5, Definition 1.7); and let $B \in Y_\psi$, $C \in Z_\psi$ and $D \in W_\psi$. Then, $\nvDash_{\mathcal{M}_M} B \wedge D$, $\nvDash_{\mathcal{M}_M} C \wedge D$ for any wr(1, 2, 3)-ms $\mathcal{M}_M$.*

Proof. (1) Immediate by Theorem 4.7. (2) By theorems 4.8, 4.9, 4.13 and Propositions 4.16 and 4.17 (cf. remarks 4.14 and 4.15). ■

5 Concluding remarks

We end the paper with some concluding remarks on the results obtained and some observations on further research to develop from them.

1. Firstly, notice that Rogerson and Restall's routes are only some particular instances of the set of general schemes falsified by one or another of the wr-ms defined in the paper (cf. Remark 3.30). To take an example of a scheme not belonging to any of the classes in Definition 1.5, consider the Generalized Modus Ponens Axiom (gMPa) $[A \wedge (A \overset{n}{\to} B)] \overset{n}{\to} B$ where

$A \overset{n}{\to} B$ is an α-formula with $A_1, ..., A_n$ being the same wff A. In [3] (pp. 72-73), Brady has shown that gMPa causes Curry's Paradox under the same circumstances as the Contraction Law W does. But in [14] (paper on which the present one is based), it is proved that gMPa is falsifiable in any wr-ms by simply using Lemma 3.11. So, a future line of research may be to use the theorems in Section 3 in order to falsify other general schemes causing Curry's Paradox but not comprehended in the classes recorded in Definition 1.5 (by the way, Rogerson and Restall suggest a similar course for furthering their own work, and the Modus Ponens Axiom $([A \wedge (A \to B)] \to B)$ is one of the theses remarked).

2. There are other (maybe many) wr-ms than those introduced by us "waiting" to be defined. Consider, for example, the wff (X) $[(p \to q) \to p] \to [(q \to p) \to p]$. X is not falsifiable in any of the general wr-ms defined above, but it is falsified in any wr(1, 2)-ms with the added clause $f_{\to}(a_F, a_F) = a_T$. Proceed as follows. Assign a_T to $\overset{1}{p}$, $\overset{3}{p}$; a_F to $\overset{2}{p}$, $\overset{4}{p}$ and b ($b \in K_2$) to q. On the other hand, notice that X is related to Peirce's Law (PL). For example, PL is immediate from X, the Permutation rule $A \to (B \to C) \Rightarrow B \to (A \to C)$, Restricted assertion $(A \to B) \to [[(A \to B) \to C)] \to C]$ and Dummett's axiom $(A \to B) \vee (B \to A)$.

3. Leaving aside Rogerson and Restall's routes, the only schemes not falsified in some of the wr-ms defined above are the following (cf. Remark 4.14, Proposition 4.16, Proposition 4.17; $\psi(p, q)$ is a positive formula): (a) Y_ψ-wffs where $n = 1$ and $deg(\psi(p, q)) > 1$; (b) Z_ψ-wffs where $n = 2$ and $deg(\psi(p, q)) > 1$; (c) W_ψ-wffs of the form $\psi(p, q) \to (p \to q)$ where $deg(\psi(p, q)) = 1$.

   Now, as it was noted in the introduction of the paper, wffs in (c) are not generally falsifiable unless we are willing to falsify some theses as the self-identity axiom $A \to A$. But theses in (a) and (b) are, however, falsifiable in most cases (actually, we have not found a particular instance not falsifiable in any w(1, 2, 3)-ms). So, let us briefly show how to proceed in general in order to falsify wffs in (a) and (b). We shall analyze the situations concerned in case (a) (those in (b) are treated similarly).

4. Y_ψ-wffs where $n = 1$ and $deg(\psi(p, q)) > 1$.

   Let $\Theta \in Y_\psi$ with $n = 1$ and $deg(\psi(p, q)) > 1$. Then, $\psi(p, q)$ is of one of the following forms: (a) $(B \to C) \to (D \to E)$; (b) $(B \to C) \to D$ with $deg(D) = 0$; (c) $D \to (B \to C)$ with $deg(D) = 0$. And we have to consider the following subcases. Θ is: (ai) $(p \to \psi(p, q)) \to \psi(p, q)$; (aii) $(\psi(p, q) \to p) \to \psi(p, q)$; (bi) $[p \to [(B \to C) \to D]] \to [(B \to C) \to D]$; (bii) $[[(B \to C) \to D] \to p] \to [(B \to C) \to D]$; (ci) $[D \to (B \to C)] \to p] \to [D \to (B \to C)]$; (cii) $[p \to [D \to (B \to C)]] \to [D \to (B \to C)]$.

Case (ai): Assign a_F to $\overset{1}{p}(A_1)$ and b ($b \in K_2$) to the rest of the variables in Θ. Then, Θ is falsified in any wr(1)-ms by Theorem 3.18.

Case (aii): Assign a_T to $\overset{0}{p}(A_1)$ and b ($b \in K_2$) to the rest of the variables in Θ. Then, Θ is falsified in any wr(2)-ms by Theorem 3.23(1).

Case (bi): Assign a_F to $\overset{1}{p}(A_1)$ and to all the variables in $\overset{0}{D}(\Theta)$ and b ($b \in K_2$) to the rest of the variables in Θ. Then, it is easy to see that Θ is falsified in any wr(1)-ms (by Theorem 3.23(1)).

Case c(i): Assign a_T to $\overset{0}{p}(A_1)$ and to all the variables in $\overset{0}{D}(\Theta)$ and ($b \in K_2$) to the rest of the variables in Θ. Then, it is easy to see that Θ is falsified in any wr(2)-ms.

Cases (bii) and (cii) are eventually shown falsifiable when pursuing with the analysis of the structure of B, C and D. We propose an example. Suppose, for instance, that $deg(D) \geq 1$ and B, C and D are positive formulas; and let M be a wr(3)-ms. Assign, then, e' to p and e to the rest of the variables in Θ. It is easy to show that this particular instance of bii is not valid in M.

Deep relevant logics have been employed by Brady in [7], or, recently, by Weber (cf. e.g., [19]) to build non-trivial naive set theories. Now, these logics (and many others) are among those verified by the wr-ms blocking the routes to triviality in Theorem 1.6 as well as many other routes, as it has been pointed out above. In this sense, we hope that the results in this paper may be of some use in establishing non-triviality results given that mere verification by a wr-ms of the propositional logic concerned guarantees that the standard routes to triviality are blocked; and given that the spectra of logics defined by each wr-ms intersects the range of logics for naive set theory amazingly extended by Brady in [8] a few months ago.

A Appendix 1: Some relevant and deep relevant logics

The following logics are formulated in the propositional language described in Definition 1.1. Firstly, we shall define Routley and Meyer's basic logic B (cf. [18], Chapter 4). The logic B can be axiomatized with the following axioms and rules

Axioms:

a1. $A \to A$

a2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

a3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

a4. $A \to (A \vee B)$ / $B \to (A \vee B)$

a5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

a6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

a7. $A \to \neg\neg A$

a8. $\neg\neg A \to A$

Rules

$$
\begin{array}{rl}
\text{Adjunction (Adj)} & A \ \& \ B \Rightarrow A \wedge B \\
\text{Modus Ponens (MP)} & A \ \& \ A \to B \Rightarrow B \\
\text{Suffixing (Suf)} & A \to B \Rightarrow (B \to C) \to (A \to C) \\
\text{Prefixing (Pref)} & B \to C \Rightarrow (A \to B) \to (A \to C) \\
\text{Contraposition (Con)} & A \to B \Rightarrow \neg B \to \neg A
\end{array}
$$

Then, we shall consider the extensions of B defined by adding to it some of the following axioms and rules (notice that a36 and a37 are not classical tautologies).

a9. $(B \to C) \to [(A \to B) \to (A \to C)]$

a10. $(A \to B) \to [(B \to C) \to (A \to C)]$

a11. $[(A \to A) \to B] \to B$

a12. $A \to [(A \to B) \to B]$

a13. $[A \to (A \to B)] \to (A \to B)$

a14. $A \to (A \to A)$

a15. $(A \to B) \to [A \to (A \to B)]$

a16. $(B \to A) \to (A \to A)$

a17. $(A \to B) \to (A \to A)$

a18. $[(A \to B) \to A] \to A$

a19. $[(A \to B) \wedge (B \to C)] \to (A \to C)$

a20. $[(A \to B) \wedge A] \to B$

a21. $[A \to (B \to C)] \to [(A \wedge B) \to C]$

a22. $(A \to B) \to [A \to (A \wedge B)]$

a23. $(A \to B) \to [(A \wedge C) \to (B \wedge C)]$

a24. $(A \to B) \to [(A \vee B) \to B]$

a25. $(A \to B) \to [(A \vee C) \to (B \vee C)]$

a26. $(A \to B) \vee (B \to A)$

a27. $A \vee (A \to B)$

a28. $A \vee \neg A$

a29. $\neg(A \wedge \neg A)$

a30. $(A \to B) \to (\neg B \to \neg A)$

a31. $(A \to \neg A) \to \neg A$

a32. $(A \wedge \neg B) \to \neg(A \to B)$

a33. $[(A \to B) \wedge \neg B)] \to \neg A$

a34. $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

a35. $\neg B \vee (A \to B)$

a36. $B \vee \neg(A \to B)$

a37. $\neg A \vee \neg(A \to B)$

Rules

$$
\begin{array}{rl}
\text{Assertion (Asser)} & A \Rightarrow (A \to B) \to B \\
\text{Specialized reductio (sr)} & A \Rightarrow \neg(A \to \neg A) \\
\text{Counterexample (Cnt)} & A \wedge \neg B \Rightarrow \neg(A \to B) \\
\text{Disjunctive Modus Ponens (MPd)} & C \vee A \ \& \ C \vee (A \to B) \Rightarrow C \vee B \\
\text{Disjunctive Suffixing (Sufd)} & C \vee (A \to B) \Rightarrow C \vee [(B \to C) \to (A \to C)]
\end{array}
$$

Disjunctive Prefixing (Prefd) $C \vee (B \to C) \Rightarrow C \vee [(A \to B) \to (A \to C)]$

Disjunctive Contraposition (Cond) $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$

Disjunctive Assertion (Asserd) $C \vee A \Rightarrow C \vee [(A \to B) \to B]$

Disjunctive specialized reductio (srd) $C \vee A \Rightarrow C \vee \neg(A \to \neg A)$

Disjunctive counterexample (Cntd) $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \to B)$

Meta-rule Summation (MRs) $A \Rightarrow B \Rightarrow C \vee A \Rightarrow C \vee B$

Deep relevant extensions of B are defined as follows:

DW: it is the result of substituting the rule Con for the corresponding axiom a30.

DJ: DW plus a19.

DK: DJ plus a28.

DR: DJ plus the rule sr.

Each of the deep relevant logics defined can "deep-relevantly" be extended with the metarule MRs.

Next, standard relevant logics can be defined as follows (the rules Suf and Pref of DW are not independent now).

TW: DW plus a9 and a10.

T: TW plus a13 and a31.

E: T plus Asser.

R: T plus a12 (a31 is not independent).

RM: R plus a14.

TW is Contractionless Ticket Entailment; T is Ticket Entailment; E, Logic of Entailment; R, Logic of Relevant Conditional, and finally, RM is R-Mingle (we remark that RM lacks the vsp: in RM the conditional $\to$ is not actually a relevant conditional. Cf. [1] and [18] about the logics defined above).

B Appendix 2: Variations on Meyer's Crystal lattice CL

In this appendix, we display particular wr-matrices upon which wr-model structures can be defined as indicated in Section 3. We exemplify each one of the wr-matrices considered in sections 2 and 3. We begin by recalling, for definiteness, the notion of a "logical matrix" as well as the standard notions related to it (in case a tester is needed, the reader can use that in [10]).

Definition B.1 (Logical matrices) *A logical matrix M is a structure* $(K, T, F, f_\to, f_\wedge, f_\vee, f_\neg)$ *where (1)* K *is a set; (2)* T *and* F *are non-empty subsets of* K *such that* $T \cup F = K$ *and* $T \cap F = \emptyset$*; (3)* $f_\to, f_\wedge$ *and* $f_\vee$ *are binary functions (distinct from each other) on* K *and* $f_\neg$ *is a unary function on* K*.*

Remark B.2 (On the set F**)** *The set* F *has been remarked in Definition B.1 only because it eases the definition of "weak relevant matrices" and "weak relevant model structures".*

In addition to Definition B.1 we set (cf. Definition 1.1):

Definition B.3 (Verification, Falsification) *Let M be a logical matrix and A a wff. (1) M verifies A iff for any assignment, v_m, of elements of K to the propositional variables of A, $v_m(A) \in T$. M falsifies A iff M does not verify A. (2) If $A_1, ..., A_n \Rightarrow B$ is a rule of derivation of a logic S, M verifies $A_1, ...A_n \Rightarrow B$ iff for any assignment, v_m, of elements of K to the variables of $A_1, ...A_n, B$, if $v_m(A_1) \in T, ..., v_m(A_n) \in T$, then $v_m(B) \in T$. M falsifies $A_1, ...A_n \Rightarrow B$ iff M does not verify it. (3) Let S be a propositional logic. M verifies S iff M verifies all axioms and rules of derivation of S.*

The matrices to follow (except the last one) can be considered as variations on the conditional characteristic of Meyer's Crystal lattice CL (wr-matrices of a different structure are displayed in [13] and [15]). The tables for $\wedge$, $\vee$ and $\neg$ are as follows (all values but 0 are designated). The structure of all matrices is:

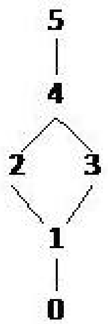

Diagram 1

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
4	4	4	4	4	4	5
5	5	5	5	5	5	5

	$\neg$
0	5
1	4
2	2
3	3
4	1
5	0

The tables for the conditional are:

$M1$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	1	2	3	4	5
2	0	0	2	0	2	5
3	0	0	0	3	3	5
4	0	0	0	0	1	5
5	0	0	0	0	0	5

$M2$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	1	2	3	4	5
2	0	0	2	0	4	5
3	0	0	0	3	4	5
4	0	0	0	0	4	5
5	0	0	0	0	0	5

$M3$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	0	2	0	2	2
3	0	0	0	3	3	3
4	0	0	0	0	1	1
5	0	0	0	0	0	1

$M4$

$\to$	0	1	2	3	4	5
0	1	1	2	3	4	5
1	0	1	2	3	4	5
2	0	0	2	0	4	5
3	0	0	0	3	4	5
4	0	0	0	0	4	5
5	0	0	0	0	0	5

$M5$

$\to$	0	1	2	3	4	5
0	1	1	2	3	4	5
1	0	1	2	3	4	4
2	0	0	2	0	2	2
3	0	0	0	3	3	3
4	0	0	0	0	1	1
5	0	0	0	0	0	1

$M6$

$\to$	0	1	2	3	4	5
0	1	1	2	3	4	5
1	0	1	2	3	4	5
2	0	0	2	0	2	5
3	0	0	0	3	3	5
4	0	0	0	0	1	5
5	0	0	0	0	0	5

$M7$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	1	2	3	4	4
2	0	0	2	0	2	2
3	0	0	0	3	3	3
4	0	0	0	0	1	1
5	0	0	0	0	0	1

We now introduce Matrix 8. The tables are as follows (all values but 0 are designated):

$M8$

$\to$	0	1	2	3	$\neg$
0	1	3	3	3	3
1	0	1	2	3	1
2	0	0	2	3	2
3	0	0	0	1	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

We record the theses and the rules (in the preceding appendix) verified by each matrix. (It has to be understood that the theses and rules omitted are falsified.)

Matrix 1 ($M1$): Meyer's Crystal lattice CL, MCL. Meyer's MCL is a simplification of Belnap's matrix M_0 used in [2] for proving for the first time that the logic of Entailment E has the vsp (M_o is also used in [1] and in [18], and it is axiomatized as well as M_{CL}, in [6]). CL verifies relevant logic R (so, it verifies the logics TW, T and E (cf. Appendix 1)). MCL verifies all rules in Appendix 1 and a9-a13, a19-a21, a27-a35. MCL is a wr(1, 2, 3)-matrix.

Matrix 2 ($M2$): M_{RMO}. M_{RMO} is a simplification of the eight-element tables used in [11] (see also [12]) to prove that the logic RMO has the vsp.

The logic RMO is axiomatized by (1) changing a30 for the corresponding rule, Con, in the formulation of R and (2) adding the axiom "mingle" (a14). M_{RMO} verifies all rules in Appendix 1 and a9-a15, a19-a21, a27-a29, a31-a35. M_{RMO} is a wr(1, 2)-matrix (it is not a wr(3)-matrix).

Matrix 3 (M3): M_{Fac}. M_{Fac} abbreviates "Matrix Factor". It is used for defining a wr-ms verifying some deep relevant extensions of B with the axiom "Factor" (a23) and related theses such as a22, in [16]. M_{Fac} verifies all extensions of B with any selection of the following axioms and rules: a13, a17, a19, a20, a21, a22, a23, a27, a28, a29, a33, a35, a37, MPd, Sufd, Prefd, Cond and MRs. M_{Fac} is a wr(1)-matrix (it is neither a w(2)-matrix, nor a wr(3)-matrix).

Matrix 4 (M4): M_{SUM}. M_{SUM} abbreviates "Matrix Summation". It is used in [16] for defining a wr-ms verifying some deep relevant extensions of B with the axiom "Summation" (a25) and related theses such as a24. M_{SUM} verifies all extensions of B with any selection of the following axioms and rules: a11, a13, a14, a15, a16, a18, a19, a20, a21, a24, a25, a27, a28, a29, a31, a32, a33, a34, a35, a36 and all rules in Appendix 1. M_{SUM} is a wr(2)-matrix (it is neither a w(1)-matrix, nor a wr(3)-matrix).

Matrix 5 (M5): $M_{\text{SUM}'}$. $M_{\text{SUM}'}$ is a modification of M_{SUM}. $M5$ verifies all extensions of DW (i.e., B plus a30 —cf. Appendix 1) with any selection of the following axioms and rules: a11, a13, a18, a19, a20, a21, a27, a28, a29, a30, a31, a32, a33, a34, a35, a36, a37, Asser, sr, Cnt, MPd, Sufd, Prefd and Cond, Asserd, Srd, Cntd. $M5$ is interesting because it can be used for defining deep relevant logics extending DR (cf. Appendix 1) with a36, a37 and other similar theses that are not classical tautologies. $M_{\text{SUM}'}$ is a wr(3)-matrix (it is neither a w(1)-matrix, nor a wr(2)-matrix).

Matrix 6 (M6): $M_{\text{SUM}''}$. $M_{\text{SUM}''}$ is also a modification of M_{SUM}. $M6$ verifies all extensions of B with any selection of the following axioms and rules: a11, a13, a18, a19, a20, a21, a27, a28, a29, a31, a32, a33, a34, a35, a36 and all rules in Appendix 1. $M_{\text{SUM}'}$ is a wr(2, 3)-matrix (it is not a w(1)-matrix).

Matrix 7 (M7): $M_{\text{Fac}'}$. $M_{\text{Fac}'}$ is a modification of M_{Fac}. $M7$ verifies all extensions of B with any selection of the following axioms and rules: a11, a13, a19, a20, a21, a27, a28, a29, a31, a32, 33, a34, a35, a37 and all rules in Appendix 1. $M_{\text{Fac}'}$ is a wr(1, 3)-matrix (it is not a w(2)-matrix).

Matrix 8 (M8). $M8$ is *a* simple four-element matrix, which is a w(1, 2, 3)-matrix. Now, let B′ be the result of changing the rule Con (cf. Appendix 1) for Con′ ($A \rightarrow B \ \& \ \neg B \Rightarrow \neg A$) in the axiomatization of B. Then, $M8$ verifies all extensions of B′ with any selection of the following axioms and rules: a11, a14, a15, a19, a20, a26, a27, a28, a29, a31, a32, a35, a36, Asser, sr, Cnt, MPd, Sufd, Prefd, Asserd and the rule Con′d. Notice that the negation defined in $M8$ is not a De Morgan negation: $\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$, $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$ and the rule Con are falsified ($v(A) = 1$ and $v(B) = 2$).

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G. Robles
Dpto. de Psicología, Sociología y Filosofía, Universidad de León
Campus de Vegazana, s/n, 24071, León, Spain
http://grobv.unileon.es
E-mail: gemmarobles@gmail.com

J. M. Méndez
Universidad de Salamanca
Campus Unamuno, Edificio FES, 37007, Salamanca, Spain
http://sites.google.com/site/sefusmendez
E-mail: sefus@usal.es