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Gemma Robles
José M. Méndez

THE BASIC CONSTRUCTIVE LOGIC FOR WEAK CONSISTENCY AND THE REDUCTIO AXIOMS

Abstract

The aim of this paper is, on the one hand, to study the effect of adding the reductio axioms to the basic constructive logics adequate to the alternative concepts of consistency defined by us. On the other hand, this is a preliminary study on the, in some way, complicated relations these logics maintain to each other.

1. Introduction

We have remarked before the peculiar effects that the reductio axioms cause when added to weak logics (cf. [7]). The aim of this paper is, on the one hand, to study the results derived from their addition to the basic constructive logics defined by us. On the other hand, it is a preliminary study on the somehow entangled relations between these constructive logics; especially entangled, indeed, when strong positive axioms are added to them. The structure of this paper is as follows. In §2, the definitions of the concepts of consistency alternative to the standard one, as well as the basic constructive logics adequate to them, are recalled. In §3, the effect of adding the (constructive) reductio axioms is studied. Finally, in §4, we provide corresponding semantic postulates to the reductio axioms in the context of B_{Kc1} (see §2), the minimal (non-positive) logic considered in this paper, and the minimal logic in the ternary relational semantics to which these axioms can be, so it seems, added if we are thinking on some kind of intuitionistic-type negation introduced with the unary connective. We note that all logics in this paper are in one, or more senses, paraconsistent logics (cf. [10]).

2. The basic constructive logic for four different concepts of consistency

Let L be a *propositional* language with a set of denumerable variables and with the connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The set of wff as well as the biconditional ($\leftrightarrow$) is defined in the usual way. By S, we refer to any logic whose language is L. The capital letters A, B, C etc. will refer to wff. Then, the notion of a theory is defined as follows:

DEFINITION 1. *A theory is a set of formulas of* L *closed under adjunction and provable entailment. That is,* Γ *is a theory iff (i) if* $A \in \Gamma$ *and* $B \in \Gamma$*, then* $A \wedge B \in \Gamma$*; and (ii) if* $A \to B$ *is a theorem of* S *and* $A \in \Gamma$*, then* $B \in \Gamma$*.*

Next, the four concepts of consistency referred to above are defined:

DEFINITION 2. (Weak consistency in a first sense) *A theory* Γ *is w1-inconsistent (weak inconsistent in a first sense) iff* $\neg A \in \Gamma$*,* A *being a theorem of* S *(a theory is w1-consistent – weak consistent in a first sense – iff it is not w1-inconsistent).*

DEFINITION 3. (Weak consistency in a second sense) *A theory* Γ *is w2-inconsistent (weak inconsistent in a second sense) iff* $A \in \Gamma$*,* $\neg A$ *being a theorem of* S *(a theory is w2-consistent – weak consistent in a second sense – iff it is not w2-inconsistent).*

DEFINITION 4. (Negation consistency) *A theory* Γ *is n-inconsistent (negation inconsistent) iff* $A \wedge \neg A \in \Gamma$ *for some wff* A *(a theory is n-consistent – negation consistent – iff it is not n-inconsistent).*

DEFINITION 5. (Absolute consistency) *A theory* Γ *is a-inconsistent (inconsistent in an absolute sense) iff* Γ *is trivial, i.e., iff every wff belongs to* Γ *(a theory is a-consistent – consistent in an absolute sense – iff it is not a-inconsistent).*

Now, the logic B_{K+} is the result of adding the K rule

$$\text{K. } \vdash A \Rightarrow \ \vdash B \to A$$

to Routley and Meyer's well-known basic positive logic B_+ (cf., e.g., [11]. The logic B_{K+} is treated with detail in [8]). Then, consider the following theses:

$$\begin{array}{rl}
\text{t1.} & \neg A \to [A \to \neg(A \to A)] \\
\text{t2.} & [B \to \neg(A \to A)] \to \neg B \\
\text{t3.} & \neg A \to [A \to (A \wedge \neg A)] \\
\text{t4.} & [B \to (A \wedge \neg A)] \to \neg B \\
\text{t5.} & (A \wedge \neg A) \to \neg(A \to A) \\
\text{t6.} & \neg A \to (\neg B \to \neg A) \\
\text{t7.} & (\neg A \wedge \neg B) \to \neg(A \vee B) \\
\text{t8.} & \neg A \to (A \to B) \\
\text{t9.} & (A \to B) \to (\neg B \to \neg A) \\
\text{t10.} & \neg B \to [(A \to B) \to \neg A] \\
\text{t11.} & (A \to \neg B) \to (B \to \neg A) \\
\text{t12.} & B \to [(A \to \neg B) \to \neg A]
\end{array}$$

Notice that t9 and t10 are the weak (constructive) contraposition axioms and t11 and t12 are the strong (constructive) contraposition axioms.

Now, the following logics are defined:

B_{Kc1}: B_{K+} plus t1 and t2.
B_{Kc4}: B_{K+} plus t3, t4 and t5.
B_{Kc6}: B_{K+} plus t2 and t8.
B_{Kc10}: B_{K+} plus t4, t6 and t7.
$B_{Kc1'}$: B_{Kc1} plus t9 and t10.
$B_{Kc4'}$: B_{Kc4} plus t9 and t10.
B_{Kc7}: B_{Kc6} plus t9 and t10.
B_{Kc2}: B_{Kc1} plus t11 and t12.
B_{Kc5}: B_{Kc4} plus t11 and t12.
B_{Kc9}: B_{Kc6} plus t11 and t12.

The logics $\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc10}}$, $\mathrm{B_{Kc4}}$ and $\mathrm{B_{Kc6}}$ are the basic constructive logics adequate to w1-consistency, w2-consistency, n-consistency and a-consistency, respectively, in the ternary relational semantics without a set of designated points (cf. [8], [10], [3] and [1]). The terms *basic*, *constructive* and *adequate* are discussed in the aforementioned papers. Then, the logics $\mathrm{B_{Kc1'}}$, $\mathrm{B_{Kc4'}}$ and $\mathrm{B_{Kc7}}$ are the results of adding the weak contraposition axioms to $\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc4}}$ and $\mathrm{B_{Kc6}}$, respectively (cf. [6], [5] and [1]); and the logics $\mathrm{B_{Kc2}}$, $\mathrm{B_{Kc5}}$ and $\mathrm{B_{Kc9}}$ are axiomatized when adding the strong constructive contraposition axioms to $\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc4}}$ and $\mathrm{B_{Kc6}}$, respectively. We remark that t9, t10 and the reductio axioms t14 and t15 (cf. §3 below) are derivable in $\mathrm{B_{Kc10}}$, and that, given the intended motivation of $\mathrm{B_{Kc10}}$, the strong contraposition axioms t11 and t12 cannot be added to it (cf. [10]).

We note:

Remark 1. *The logic* $\mathrm{B_{Kc3}}$ *is the result of adding* t8, t11 *and* t12 *to* $\mathrm{B_{Kc1}}$. *Then, we note that* $\mathrm{B_{Kc3}}$ *and* $\mathrm{B_{Kc9}}$ *are deductively equivalent (cf. [8]). The logic* $\mathrm{B_{Kc8}}$ *is axiomatized by replacing* t8 *by* t13 $A \to (\neg A \to B)$ *in* $\mathrm{B_{Kc6}}$. *Finally, the logics* $\mathrm{B_{Kc1'}}$ *and* $\mathrm{B_{Kc4'}}$ *are defined in [6] and [5], respectively.*

Remark 2. *All logics except* $\mathrm{B_{Kc10}}$ *can equivalently be defined with a falsity constant instead of the unary connective (cf. [9], [2], [4], [10]).*

The relations that the logics defined above maintain to each other can be summarized in the following diagram (the arrow $\to$ stands for $\supseteq$). That these are the only relations that can be obtained between these logics is proved, on the one hand, in [8], [3], [1] and [10], and, on the other hand, by using MaGIC, the matrix generator developed by J. Slaney (cf. [13]). (Cf. Appendix).

Diagram 1

3. Adding the (constructive) reductio axioms to the basic constructive logics

The constructive reductio axioms are

$$\text{t14. } (A \to \neg B) \to [(A \to B) \to \neg A]$$

and

$$\text{t15. } (A \to B) \to [(A \to \neg B) \to \neg A]$$

Now, the aim of this note is to study the effect of adding t14 and t15 to the basic constructive logics $\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc4}}$ and $\mathrm{B_{Kc6}}$. We note:

REMARK 3. *The following are theorems of* $\mathrm{B_{Kc1}}$ *(the first one is a theorem of* $\mathrm{B_{K+}}$ *(cf. [11]):*

$$\text{t16. } (A \to B) \to [A \to (A \wedge B)]$$

$$\text{t17. } \neg A \to (A \to \neg B)$$

Then, we have:

PROPOSITION 1. *Let* $\mathrm{B_{K+\neg}}$ *be any negation extension of* $\mathrm{B_{K+}}$ *with* t10 *and* t18 $(A \to \neg B) \to \neg(A \wedge B)$ *(respectively,* t19 $(A \to B) \to \neg(A \wedge \neg B)$*). Then,* t14 *(respectively,* t15*) is a theorem of* $\mathrm{B_{K+\neg}}$*.*

PROOF. We prove that t14 is a theorem of the given logic $\mathrm{B_{K+,\neg}}$ (the other part of the proof is similar).

1.	$\neg(A \wedge B) \to \{[A \to (A \wedge B)] \to \neg A\}$	t10
2.	t14	t16, t18, 1

□

PROPOSITION 2. *Let* $\mathrm{B_{K+\neg}}$ *be any negation extension of* $\mathrm{B_{K+}}$ *with* t14 *(or with* t15*) and* t17*. Then,* t6, t9 *and* t10 *are theorems of* $\mathrm{B_{K+\neg}}$*.*

PROOF. We prove that t6, t9 and t10 are theorems of the given logic $\mathrm{B_{K+,\neg}}$ (the other part of the proof is similar).

1.	$(A \to \neg A) \to [(A \to A) \to \neg A]$	T14
2.	$\neg A \to [(A \to B) \to \neg A]$	1, t17
3.	$\neg A \to (B \to \neg A)$	2, K
4.	$\neg B \to [(A \to B) \to \neg A]$	t14, 3
5.	$(A \to B) \to \neg(A \wedge \neg B)$	t14
6.	$\neg B \to [A \to (A \wedge \neg B)]$	t16, 3
7.	$(A \to B) \to (\neg B \to \neg A)$	4, 5, 6

□

PROPOSITION 3. *Let* $\mathrm{B_{K+\neg}}$ *be any negation extension of* $\mathrm{B_{K+}}$ *with* t4*. Then,* t18 *and* t19 *are theorems of* $\mathrm{B_{K+\neg}}$*.*

PROOF. We prove that t18 is derivable (the proof for t19 is similar). First it is proved $\vdash A \to B \Rightarrow \vdash (A \to \neg B) \to \neg A$:

1.	$\vdash A \to B$	Hyp.
2.	$\vdash (A \to \neg B) \to [(A \to B) \wedge (A \to \neg B)]$	1, K
3.	$\vdash [(A \to B) \wedge (A \to \neg B)] \to \neg A$	t4, $\mathrm{B_{K+}}$
4.	$\vdash (A \to \neg B) \to \neg A$	2, 3

Then, t18 is immediate. □

As a corollary of Proposition 1 and 3, we have:

PROPOSITION 4. *The reductio axioms* t14 *and* t15 *are theorems of* $\mathrm{B_{Kc4'}}$.

Now, let $\mathrm{B_{Kcn}}$ be any of the constructive logics treated in this paper, and tm any of the axioms here considered. The logic $\mathrm{B_{Kcntm}}$ is the result of adding tm to $\mathrm{B_{Kcn}}$. Thus, for example, $\mathrm{B_{Kc4t14}}$ (respectively, $\mathrm{B_{Kc4t15}}$) is the extension of $\mathrm{B_{Kc4}}$ with t14 (respectively, t15). Now, from Proposition 2 follows immediately:

PROPOSITION 5. *The weak contraposition axioms* t9 *and* t10 *are theorems of* $\mathrm{B_{Kc4t14}}$ *and* $\mathrm{B_{Kc4t15}}$.

And, as a corollary of Propositions 4 and 5:

PROPOSITION 6. *The logics* $\mathrm{B_{Kc4'}}$, $\mathrm{B_{Kc4t14}}$ *and* $\mathrm{B_{Kc4t15}}$ *are deductively equivalent.*

Next, it is proved:

PROPOSITION 7. *The logic* $\mathrm{B_{Kc1t14}}$ *(*$\mathrm{B_{Kc1t15}}$*) is deductively equivalent to* $\mathrm{B_{Kc4'}}$.

PROOF. (a) Given Proposition 2, we only have to prove that t3, t4 and t5 are theorems of $\mathrm{B_{Kc1t14}}$ ($\mathrm{B_{Kc1t15}}$). We prove that this is the case in respect of $\mathrm{B_{Kc1t14}}$ (the other part of the proof is similar).

1.	$\neg A \to [A \to (A \wedge \neg A)]$	t16, t17
2.	$\neg(A \wedge \neg A)$	t14
3.	$B \to \neg(A \wedge \neg A)$	2, K
4.	$[B \to (A \wedge \neg A)] \to \neg B$	t14, 3
5.	$(A \wedge \neg A) \to \neg B$	2, t17

(b) As $\mathrm{B_{Kc1}}$ is included in $\mathrm{B_{Kc4}}$ (cf. [3]), the proof in the inverse direction follows by Proposition 4. □

As a corollary of Proposition 6 and 7, we have:

PROPOSITION 8. *The following logics are deductively equivalent:* $\mathrm{B_{Kc4'}}$, $\mathrm{B_{Kc4t14}}$, $\mathrm{B_{Kc4t15}}$, $\mathrm{B_{Kc1t14}}$ *and* $\mathrm{B_{Kc1t15}}$.

Now, given that the EFQ ('E falso quodlibet') axiom t8 is not derivable in any logic equivalent to, or included in, B_{Kc5} (notice that B_{Kc5} is a sublogic of minimal intuitionistic logic) the following relations are obtained between the basic constructive logics when the reductio axioms are added:

Diagram 2

PROOF. The proof is by Proposition 8, Diagram 1, and by using MaGIC again, when needed. □

Finally, we note the following:

REMARK 4. *(a) The strong contraposition axioms are not derivable in* B_{Kc6t14} *(so, in none of the logics included in it, cf. Diagram 2). (b) The reductio axioms, derivable in* B_{Kc5}*, are not derivable in* B_{Kc9} *(so, neither are they in* B_{Kc2}*). (c) If the reductio axioms are added to* B_{Kc2} *and* B_{Kc9}*, the resulting relations between these systems are:*

Diagram 3

Results in a and b are by MaGIC, and that in c is obvious.

4. Semantics for the reductio axioms in the context of $\mathbf{B_{Kc1}}$

We provide a semantics for the constructive reductio axioms in the context of the logic $\mathrm{B_{Kc1}}$. Knowledge of the Routley-Meyer ternary relational semantics for relevant logics (cf., e.g., [11]), as well as that of the ternary semantics for $\mathrm{B_{Kc1}}$ (cf. [8]) is presupposed. Consider the following semantic postulates:

P1. $R^2abcd \ \& \ d \in S \Rightarrow (\exists x, y \in K)(\exists z \in S)(Racx \ \& \ Rbcy \ \& \ Rxyz)$

P2. $R^2abcd \ \& \ d \in S \Rightarrow (\exists x, y \in K)(\exists z \in S)(Racx \ \& \ Rbcy \ \& \ Ryxz)$

We shall prove:

Proposition 9. P1 *and* P2 *are the corresponding postulates (c.p) to, respectively,* t14 *and* t15. *That is, given* $\mathrm{B_{Kc1}}$*-semantics,* t14 *(respectively,* t15*) is proved valid with* P1 *(respectively,* P2*). And given the logic* $\mathrm{B_{Kc1}}$, P1 *(respectively,* P2*) is proved canonically valid with* t14 *(respectively,* t15*).*

In order to prove Proposition 9, we note:

Remark 5. *(Cf. [8]) (a) The following are rules of* $\mathrm{B_{Kc1}}$*:* t20 $\vdash A \Rightarrow \vdash \neg\neg A$*;* t21 $\vdash A \Rightarrow \vdash (B \to \neg A) \to \neg B$*. (b)* $R^C abc \Rightarrow b \subseteq c$ *holds in the* $\mathrm{B_{Kc1}}$ *canonical model (in fact, in the* $\mathrm{B_{K+}}$ *canonical model).*

Next, we prove that P2 is the c.p to t15 (the proof for P1 and t14 is similar).

PROOF. (a) t15 *is valid:* Suppose that t15 is not valid. Then, $a \vDash A \to B$, $a \nvDash (A \to \neg B) \to \neg A$ for some $a \in K$ in some model. So, for b, $c \in K$ such that $Rabc$, $b \vDash A \to \neg B$, $c \nvDash \neg A$. Consequently, for certain $d \in K$ and $e \in S$ such that $Rcde$, $d \vDash A$. As $Rabc$ and $Rcde$, we have R^2abde. As $e \in S$, $Radx$, $Rbdy$, $Ryxz$ follow for some x, $y \in K$ and $z \in S$ in this model (P2). Now, $x \vDash B$ ($Radx$ and $d \vDash A$) and $y \vDash \neg B$ ($Rbdy$, $b \vDash A \to \neg B$ and $d \vDash A$). So, $x \nvDash B$ ($y \vDash \neg B$, $Ryxz$ and $z \in S$), contradicting $x \vDash B$ above.

(b) P2 *is canonically valid:* Suppose $R^{C2}abcd$ and $d \in S^C$. That is, suppose R^Cabu and R^Cucd for some $u \in K^C$ and $d \in S^C$. We have to prove that there are x, $y \in K^C$ and $z \in S^C$ such that R^Cacx, R^Cbcy and R^Cyxz. So, define the theories $x = \{B \mid \exists A[A \to B \in a \;\;\&\;\; A \in c]\}$, $y = \{B \mid \exists A[A \to B \in b \;\;\&\;\; A \in c]\}$ and $z = \{B \mid \exists A[A \to B \in y \;\;\&\;\; A \in x]\}$. We see that R^Tacx, R^Tbcy and R^Tyxz. We have to prove that z is w1-consistent (cf. Definition 2). So, for *reductio ad absurdum*, suppose that $\neg A \in z$, A being a theorem of $\mathrm{B_{Kc1}}$ plus t15. By definition of x, y, z, $C \to (B \to \neg A) \in b$, $D \to B \in a$ for some wff B and wffs $C \in c$, $D \in c$. By t21 (Remark 5a), $\vdash (B \to \neg A) \to \neg B$ is a theorem. So, $[C \to (B \to \neg A)] \to (C \to \neg B)$ is also a theorem, and, consequently, $C \to \neg B \in b$. By R^Cabu, $b \subseteq u$ follows (Remark 5b). So, $C \to \neg B \in u$. Then, $\neg B \in d$ (R^Cucd, $C \in c$). On the other hand, as any theory contains all theorems of $\mathrm{B_{Kc1t15}}$, $D \to D \in u$; and, then, $D \in d$ (R^Tucd, $D \in c$). Consequently, $D \wedge \neg B \in d$.

Next, we prove that $\neg(D \wedge \neg B) \in d$, whence the w1-inconsistency of z is untenable. By t19, $\vdash (D \to B) \to \neg(D \wedge \neg B)$. Then, $\neg(D \wedge \neg B) \in a$ ($D \to B \in a$). Now, let E be a theorem (note that $\neg\neg E$ is also a theorem by t20 —Remark 5a—). By t17, $\neg(D \wedge \neg B) \to [(D \wedge \neg B) \to \neg E]$. So, $(D \wedge \neg B) \to \neg E \in a$ whence $\neg\neg E \to \neg(D \wedge \neg B) \in a$ by t9. As $\neg\neg E \in b$ ($\neg\neg E$ is a theorem), $\neg(D \wedge \neg B) \in u$ (R^Cabu). Again, by t17 and t9, $\neg\neg E \to \neg(D \wedge \neg B) \in u$, and given that $\neg\neg E \in c$, $\neg(D \wedge \neg B) \in d$ (R^Cucd). Consequently, $[(D \wedge \neg B) \wedge \neg(D \wedge \neg B)] \in d$. By t15, $\neg(A \wedge \neg A)$ is immediate. So, $\neg[(D \wedge \neg B) \wedge \neg(D \wedge \neg B)]$ is a theorem, whence $[(D \wedge \neg B) \wedge \neg(D \wedge \neg B)] \to \neg E$ is also a theorem by t17 (cf. Remark 3). Therefore, d contains every negation formula, which contradicts its w1-consistency. Finally, x, y and z are extended to the required prime theories in the standard way (cf. [8]).□

To end the paper, we note that (a) the strong contraposition axioms t11 and t12 have not been necessary in the proof of the canonical validity of P2, and that (b) given the soundness and completeness of $\mathrm{B_{Kc1}}$, we have in fact provided a semantics for B_{Kc1t14} (B_{Kc1t15}) independent of that for B_{Kc4} and its extensions.

Appendix

The following sets of matrices have been found by MaGIC (cf. [13]). Each one of them satisfies the axioms and rules of B_+ and the K rule. Designated values are starred. MS abbreviates "Matrix Set".

MSI:

→	0	1	2
0	2	2	2
1	1	2	2
*2	0	0	2

∧	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

∨	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

a)

	¬
0	2
1	2
*2	2

b)

	¬
0	2
1	1
*2	0

Both matrices satisfy t1 and t2. So, the positive matrices plus (a) (MSIa) or plus (b) (MSIb) verify B_{Kc1}. Now, MSIa falsifies t8 ($A = 1$, $B = 0$). So, B_{Kc1} does not include B_{Kc6}. MSIb falsifies t4 ($A = B = 1$), t6 ($A = 1$, $B = 2$) and t9 ($A = 1$, $B = 0$). So, B_{Kc1} includes neither B_{Kc4} nor B_{Kc6} nor $B_{Kc1'}$. Consequently, B_{Kc1} includes none of the logics treated in the paper (except, obviously, itself and the positive logics B_+ and B_{K+}).

MSII:

→	0	1	2	3
0	3	3	3	3
1	1	3	3	3
2	1	1	3	3
*3	0	1	1	3

∧	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
*3	0	1	2	3

∨	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
*3	3	3	3	3

a)

	¬
0	3
1	3
2	1
*3	1

b)

	¬
0	3
1	1
2	1
*3	0

MSIIa and MSIIb verify $\mathrm{B_{Kc1'}}$ (they satisfy t1, t2, t9, t10). MSIIa falsifies t8 ($A = 1$, $B = 0$). So, $\mathrm{B_{Kc1'}}$ does not include $\mathrm{B_{Kc6}}$. MSIIb falsifies t4 ($A = B = 1$) and t11 ($A = 1$, $B = 2$). So, $\mathrm{B_{Kc1'}}$ includes neither $\mathrm{B_{Kc4}}$ nor $\mathrm{B_{Kc2}}$. Consequently, no logic in the paper except $\mathrm{B_{Kc1}}$ is included in $\mathrm{B_{Kc1'}}$.

MSIII:

$\to$	0	1	2	3	4
0	4	4	4	4	4
1	0	4	0	4	4
2	1	1	4	4	4
3	0	1	0	4	4
*4	0	0	0	0	4

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
*4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	4	3	4
2	2	4	2	4	4
3	3	3	4	3	4
*4	4	4	4	4	4

a)

	$\neg$
0	4
1	4
2	4
3	4
*4	4

b)

	$\neg$
0	4
1	0
2	1
3	0
*4	0

MSIIIa and MSIIIb verify $\mathrm{B_{Kc4}}$ (they satisfy t3, t4 and t5). MSIIIa falsifies t8 ($A = 1$, $B = 0$). So, $\mathrm{B_{Kc4}}$ does not include $\mathrm{B_{Kc6}}$. MSIIIb falsifies t6 ($A = 2$, $B = 0$) and t9 ($A = 2$, $B = 0$). So, $\mathrm{B_{Kc4}}$ includes neither $\mathrm{B_{Kc10}}$ nor $\mathrm{B_{Kc1'}}$. Consequently, no logic in the paper except $\mathrm{B_{Kc1}}$ is included in $\mathrm{B_{Kc4}}$.

MSIV:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	0	2	2	2
*2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

MSIV verifies $\mathrm{B_{Kc10}}$ (it satisfies t4, t6 and t7) and falsifies t1 ($A = 1$) and t5 ($A = 1$). So, $\mathrm{B_{Kc10}}$ includes neither $\mathrm{B_{Kc1}}$ nor $\mathrm{B_{Kc4}}$, and, consequently, it includes none of the other logics treated in the paper.

MSV:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	2	3	3	3	2
2	0	0	3	3	0
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
*3	3	3	3	3

MSV verifies $\mathrm{B}_{\mathrm{Kc6}}$ (it satisfies t2 and t8) and falsifies t4 ($A = B = 1$), t13 ($A = B = 1$) and t9 ($A = 1$, $B = 0$). So, $\mathrm{B}_{\mathrm{Kc6}}$ includes neither $\mathrm{B}_{\mathrm{Kc4}}$ nor $\mathrm{B}_{\mathrm{Kc1'}}$. Consequently, no logic in the paper except $\mathrm{B}_{\mathrm{Kc1}}$ is included in $\mathrm{B}_{\mathrm{Kc6}}$.

MSVI:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
*2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

MSVI verifies $\mathrm{B}_{\mathrm{Kc8}}$ (it satisfies t2 and t13) and falsifies t4 ($A = B = 1$), t6 ($A = 1$, $B = 2$) and t9 ($A = 1$, $B = 0$). So, $\mathrm{B}_{\mathrm{Kc8}}$ includes neither $\mathrm{B}_{\mathrm{Kc4}}$ nor $\mathrm{B}_{\mathrm{Kc1'}}$ nor $\mathrm{B}_{\mathrm{Kc10}}$. Consequently, no logic in the paper except $\mathrm{B}_{\mathrm{Kc6}}$ is included in $\mathrm{B}_{\mathrm{Kc8}}$.

MSVII:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	1
2	1	1	3	3	1
*3	0	1	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
*3	3	3	3	3

MSVII verifies $\mathrm{B}_{\mathrm{Kc7}}$ (it satisfies t2, t8, t9 and t10) and falsifies t4 ($A = B = 1$), t11 ($A = 1$, $B = 2$) and t13 ($A = 2$, $B = 0$). Consequently, $\mathrm{B}_{\mathrm{Kc7}}$ does not include $\mathrm{B}_{\mathrm{Kc4}}$ (so it does not include $\mathrm{B}_{\mathrm{Kc10}}$ nor $\mathrm{B}_{\mathrm{Kc4'}}$, $\mathrm{B}_{\mathrm{Kc5}}$ either) $\mathrm{B}_{\mathrm{Kc2}}$ (so, neither does it include $\mathrm{B}_{\mathrm{Kc9}}$) and $\mathrm{B}_{\mathrm{Kc8}}$.

MSVIII:

$\to$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	1	1	3	3
*3	0	1	0	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
*3	3	3	3	3

a)

	$\neg$
0	3
1	0
2	1
*3	0

b)

	$\neg$
0	3
1	3
2	1
*3	1

MSVIIIa and MSVIIIb verify $B_{Kc4'}$ (they satisfy t3, t4, t5, t9 and t10). MSVIIIa falsifies t11 ($A = 1$, $B = 2$). MSVIIIb falsifies t8 ($A = 1$, $B = 0$). Consequently, $B_{Kc4'}$ does not include B_{Kc2} (so, neither does it include B_{Kc5}, B_{Kc9}) and B_{Kc6} (so, neither does it include B_{Kc7}, B_{Kc8}).

MSIX:

$\to$	0	1	2
0	2	2	2
1	1	2	2
*2	0	1	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

a)

	$\neg$
0	2
1	1
*2	0

b)

	$\neg$
0	2
1	2
*2	1

MSIXa and MSIIXb verify B_{Kc2} (they satisfy t1, t2, t11and t12). MSIXa falsifies t4 ($A = B = 1$). MSVIIIb falsifies t8 ($A = 1$, $B = 0$). Consequently, B_{Kc2} does not include B_{Kc4} (so, neither does it include B_{Kc10}, $B_{Kc4'}$ and B_{Kc5}) and B_{Kc6} (so, neither does it include B_{Kc7}, B_{Kc8} and B_{Kc9}).

MSX:

$\to$	0	1	$\neg$
0	1	1	1
*1	0	1	1

$\wedge$	0	1
0	0	0
*1	0	1

$\vee$	0	1
0	0	1
*1	1	1

MSX verifies B_{Kc5} (it satisfies t3, t4, t5 and t12) and falsifies t8 ($A = 1$, $B = 0$). Consequently, B_{Kc5} does not include B_{Kc6} (so, neither does it include B_{Kc7}, B_{Kc8} and B_{Kc9}).

MSXI:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
*2	0	1	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

MSXI verifies B_{Kc9} (it satisfies t1, t8, t11 and t12) and falsifies t4 ($A = B = 1$). Consequently, B_{Kc9} does not include B_{Kc4} (so, neither does it include B_{Kc10}, $B_{Kc4'}$ and B_{Kc5}).

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Dpto. de Historia y Filosofía de la CC, la Ed. y el Leng.
Universidad de La Laguna
Facultad de Filosofía, Campus de Guajara
38071, La Laguna, Tenerife, Spain
http://webpages.ull.es/users/grobles
e-mail: gemmarobles@gmail.com

Universidad de Salamanca
Campus Unamuno, Edificio FES
37007, Salamanca, Spain
http://web.usal.es/˜sefus
e-mail: sefus@usal.es