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THE LOGIC B AND THE REDUCTIO AXIOMS

Abstract

We study the possibilities of introducing the reductio axioms in Routley's basic logic B. We show how to define special reductio and conjecture that full reductio cannot be introduced. Complete relational ternary semantics are provided for all the logics in the paper.

1. Introduction

In [2], we showed how to introduce a constructive negation defined with a falsity constant F in B+. The result is the logic called Bmr. This constructive negation can be characterized by the following theses: weak double negation [$A \to \neg\neg A$], weak special reductio [$(A \to \neg A) \to \neg A$], weak contraposition [$(A \to B) \to (\neg B \to \neg A)$ and $(A \to \neg B) \to (B \to \neg A)$]. Along that paper we discussed the possibilities of introducing the (weak) full reductio axiom wra [$(A \to B) \to [(A \to \neg B) \to \neg A]$] in Bmr (see §6 below). The aim of this paper is to investigate the possibilities of introducing the (weak and strong) full reductio axioms (wra and $(\neg A \to \neg B) \to [(\neg A \to B) \to A]$) in the logic B . As it is known, B is the result of adding strong double negation and contraposition as a rule to the basic positive logic B+ (see, e.g., [1] or [3])

In particular, the structure of the paper is as follows. In §1, we recall the logic B. In §2, we define the logic Bcon (B plus the contraposition axioms). In §3, we define the logic Bconr (Bcon plus the special reductio axioms) and prove some syntactical and semantical facts about it. In §4, we prove that the full reductio axioms are not derivable in Bconr, and in §5, we discuss the possibilities of introducing the full reductio axioms in Bconr. Finally, in §6, we draw some conclusions from §5 via the definition of two stronger logics with reductio.

As it is known, relevant negation is (syntactically) axiomatized with double negation, contraposition and reductio (or semantically, in the ternary relational semantics, with P1, P3 and P4. See §2-§4 below). We think that our paper clears up the limits to introduce relevant negation (within the context of the ternary relational semantics) in weak positive logics. Complete semantics are defined for all the logics in the paper. We assume acquaintance with the ternary relational semantics and, in particular, with the logics B+ and B.

2. The logic B

The logic B is the result of adding the following axiom and the rule of inference contraposition (con) to B+:

A1. $\neg\neg A \to A$

Con. if $\vdash A \to \neg B$, then $\vdash B \to \neg A$

We note that T1-T5 and R1-R3 below are derivable in B:

T1. $A \to \neg\neg A$

T2. $\neg (A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

T3. $\neg (A \wedge B) \leftrightarrow (\neg A \vee \neg B)$

T4. $(A \vee B) \leftrightarrow \neg (\neg A \wedge \neg B)$

T5. $(A \wedge B) \leftrightarrow \neg (\neg A \vee \neg B)$

R1. $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$

R2. $\vdash \neg A \to B \Rightarrow \vdash \neg B \to A$

R3. $\vdash \neg A \to \neg B \Rightarrow \vdash B \to A$

B models are defined adding the operation * ("Routley Star") to B+ models together with the valuation clause and the postulates below:

(v¬) $a \models \neg A$ iff $a* \not\models A$

P1. $a \Leftrightarrow a**$

P2. $a \leq b \Rightarrow b* \leq a*$

A formula A is valid iff $a \models A$ for all $a \in O$ in all models (O is a selected subset of the set of possible worlds K).

3. The logic Bcon

The logic Bcon is defined adding to B the axiom

A2. $(A \to \neg B) \to (B \to \neg A)$

In addition to T1-T5, we have the following theorems:

T6. $(A \to B) \to (\neg B \to \neg A)$
T7. $(\neg A \to B) \to (\neg B \to A)$
T8. $(\neg A \to \neg B) \to (B \to A)$

A Bcon model is the same as a B model save for the substitution of P2 (which now becomes derivable) for

P3. $Rabc \Rightarrow Rac * b*$

A2 is valid by P3 and, on the other hand, it is clear that in order to prove the completeness of Bcon, we just have to prove that P3 holds canonically (see, for example, in [1] or [3] how P3 is proved canonically valid with T6).

4. The logic Bconr

The logic Bconr is axiomatized addding the following axiom to Bcon:

A3. $(A \to \neg A) \to \neg A$

Then, all forms of reductio as a rule are derivable

R4. $\vdash A \to B \Rightarrow \vdash (A \to \neg B) \to \neg A$	A2, A3
R5. $\vdash A \to \neg B \Rightarrow \vdash (A \to B) \to \neg A$	R4, T1
R6. $\vdash \neg A \to B \Rightarrow \vdash (\neg A \to \neg B) \to \neg A$	R4, A1
R7. $\vdash \neg A \to \neg B \Rightarrow \vdash (\neg A \to B) \to A$	R6, T1
R8. $\vdash A \to B \Rightarrow \vdash (\neg A \to B) \to B$	R7, T6, T7
R9. $\vdash \neg A \to B \Rightarrow \vdash (A \to B) \to B$	R8, A1

and in addition to T1-T8, the following theorems:

T9. $(A \to \neg B) \to \neg (A \wedge B)$	R4
T10. $(A \to B) \to \neg (A \wedge \neg B)$	T1, T9
T11. $(A \to B) \to (\neg A \vee B)$	A1, T3, T10
T12. $(\neg A \to B) \to (A \vee B)$	A1, T11
T13. $(\neg A \to A) \to A$	A1, A3, T1
T14. $\neg (A \wedge \neg A)$	T10
T15. $\neg A \vee A$	T11

We note the following:

PROPOSITION 1. *Any of the rules R4-R9 and theorems T9-T13 can axiomatize Bconr instead of A3*

PROOF. We sketch a proof of proposition 1:

1. R4-R9 are equivalent given Bcon: they are interdeducible by contraposition and double negation.

2. Regarding A3 and R4-R9:

(a) A3 is immediately derivable from R7. Therefore, by (i), it is provable from any of the rules R4-R9.

(b) Given Bcon, R4-R9 are derivable from A3. Prove, e.g., R4 with A3 and A2.

3. A3 and T9-T13 are equivalent given Bcon.

(a) Prove T9 with R4 . Hence, T9 is derivable from A3 by (ii)(b)

(b) Prove A3 with T9

(c) T9-T12 are equivalent by contraposition, double negation and De Morgan laws. Hence, A3 is derivable from T9-T12 by (iii)(b).

4. A3 is derivable from T13 by A1 and T1. □

We note that T14 and T15 cannot axiomatize Bconr instead of A3.

A Bconr model is the same as a Bcon model but with the addition of the postulate

P4. $Raa * a$

We note that the following postulate is immediately derivable in all Bconr models with P3:

P5. $Ra * aa*$

A3 is valid by P4, and P4 is canonically valid by T10 (see, for example, [1] or [3]). Moreover, in connection with Proposition 1, we have the following:

PROPOSITION 2. *Given Bcon, the corresponding semantic postulate for A3, R4-R9, T9-T13 is P4.*

The "corresponding semantic postulate" is defined as follows:

DEFINITION 1. Pi is the corresponding semantic postulate for Ti iff (i) Ti is valid by Pi and (ii) Pi is proved canonically valid with Ti.

PROOF. In order to prove Proposition 2, prove that P4 is canonically valid with T10. Next, use Proposition 1. □

5. The full reductio axioms are not derivable in Bconr

Consider the following set of matrices (2 and 3 are designated values)

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	2	2	3	2
2	0	1	2	3	1
3	0	0	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

This set verifies Bconr but falsifies full reductio, to wit,

t1. $(A \to B) \to [(A \to \neg B) \to \neg A] \quad (A = 1, B = 3)$
t2. $(A \to \neg B) \to [(A \to B) \to \neg A] \quad (A = 1, B = 0)$
t3. $(\neg A \to B) \to [(\neg A \to \neg B) \to A] \quad (A = 3, B = 2)$
t4. $(\neg A \to \neg B) \to [(\neg A \to B) \to A] \quad (A = 0, B = 2)$
t5. $(A \to B) \to [(\neg A \to B) \to B] \quad (A = 0, B = 2)$
t6. $(\neg A \to B) \to [(A \to B) \to B] \quad (A = 3, B = 2)$

6. On the introduction of the full reductio axioms

We prove the

PROPOSITION 3. *Consider the following semantic postulates:*
p1. $Rabc \Rightarrow \exists x\,[Rbc * x \& Rac * x*]$
p2. $Rabc \Rightarrow \exists x\,[Rac * x \& Rbc * x*]$
p3. $Rabc \Rightarrow \exists x\,[Rax * c \& Rbxc]$
p4. $Rabc \Rightarrow \exists x\,[Raxc \& Rbx * c]$
Let us add p1-p4 to Bconr models. Now, t1 and t2 are valid by p1 (or p2), t3 and t4 by p2 (or p1), and t5 and t6 by p3 (or p4).

PROOF. Let us prove, for example, that t5 is valid (the rest of the cases are similar and are left to the reader).

Suppose for some $a \in K$ $a \models A \to B$, $a \not\models (\neg A \to B) \to B$. By definitions $Rabc$, $b \models \neg A \to B$, $c \not\models B$ for some b, $c \in K$. Now, by p3, $Rax * c$, $Rbxc$ for some x in K. Then, from $Rbxc$, $b \models \neg A \to B$ and $c \not\models B$ it follows $x \not\models \neg A$, i.e., $x* \models A$. Hence, $c \models B$ by $Rax * c$, $a \models A \to B$ and

$x* \models A$. Our initial supposition leading to a contradiction, we conclude t5 is valid (we remark that, given Bcon models, p1 and p4 (p2 and p3) are equivalent). $\Box$

But, what about the canonical adequacy of p1-p3? Let us try the proof of, for example, p1. That is, we wish to prove

PROPOSITION 4. *The canonical p1, i.e.,* $R^C abc \Rightarrow \exists x[R^C bc * x \ \& \ R^C ac * x*]$ *holds in the canonical model.*

PROOF. Suppose for $a, b, c \in K^C$, $R^C abc$. Define $x = \{B : \exists A(A \in c* \ \& \ A \to B \in b)\}$. Then, $R^C ac * x*$ is proved as follows. Let $A \to B \in a$, $a \in c*$. We have to prove that $B \in x*$. Suppose, on the contrary, $B \notin x*$. Then, $\neg B \in x$. By definition of x, $D \to \neg B \in b$ for some $D \in c*$. By contraposition, $B \to \neg D \in b$. Consider now the thesis *suffixing* $\vdash (A \to B) \to [(B \to \neg D) \to (A \to \neg D)]$. *If suffixing is a theorem*, then $(B \to \neg D) \to (A \to \neg D) \in a$. Now, given $R^C abc$ and $R^C cc * c$ (P4), $\mathrm{R}^{2C} abc * c$, i.e., $R^C abz$ and $R^C zc * c$ for some $z \in K^C$. By $R^C abz$, $(B \to \neg D) \to (A \to \neg D) \in a$ and $B \to \neg D \in b$, we get $A \to \neg D \in z$. Applying contraposition, $D \to \neg A \in z$. Therefore, by $R^C zc * c$ and $D \in c*$, $\neg A \in c$, i.e., $A \notin c*$, a contradiction. Consequently, we have $R^T bc * x$ and $R^T ac * x*$. Finally, x is extended to a prime theory y such that $R^C bc * y$ and $R^C ac * y*$ as required. $\Box$

In Proposition 4, we said "[...] holds in the canonical model". But, which canonical model? In the proof of Proposition 4, the sentence "if suffixing is a theorem" is emphasized. And here is the point. On the one hand, suffixing is not a theorem of Bconr (it is falsified by the matrix in §4 when $A = 1$, $B = 0$, $C = 0$). But, on the other hand, it is our conjecture that suffixing or some thesis related to it is needed in the proof of the canonical adequacy of P4. Whence, according to this conjecture, full reductio cannot be introduced in Bconr. Anyway, we establish in the following section a setting to discuss the point.

7. The logics BpconR and BsconR

Consider the axiom

A10. $(B \to C) \to [(A \to B) \to (A \to C)]$

Let Bpcon (Bcon plus prefixing) be the result of adding A10 to Bcon. We remark that Bscon (Bcon plus suffixing) i.e., Bcon plus the following axiom is an equivalent system (A10 and this axiom are equivalent by contraposition):

A11. $(A \to B) \to [(B \to C) \to (A \to C)]$

A Bpcon model is the same as a Bcon model but with the addition of the postulate

P6. $R^2abcd \Rightarrow \exists x\,(Rbcx \& Raxc)$

On the other hand, a Bscon model is the same as a Bcon model but with the addition of the postulate

P7. $R^2abcd \Rightarrow \exists x\,(Racx \& Rbxd)$

Now, P6 and P7 are the corresponding semantic postulates for A10 and A11 (see [1] or [3]). So, Bpcon and Bscon are complete in respect of Bpcon models and Bscon models, respectively. We now define the logics BpconR and BsconR in what follows.

The logic BpconR (BsconR) is the result of adding any of the full reductio axioms (t1-t6) in §4. That is to say, BpconR (BsconR) is the result of adding full reductio axioms to Bpcon (Bscon). (Note that, given Bcon, t1-t6 are equivalent).

A BpconR model (BsconR model) is the result of adding the semantic postulate P6 (P7) to Bconr models. We remark

PROPOSITION 5. *The postulates p1-p4 are derivable in BpconR models and BsconR models.*

PROOF. Let us prove, for example, that p1 is derivable. Suppose $Rabc$. Given $Rcc*c$ (P4), R^2abc*c. By P7, $Rbc*x$ and $Raxc$ for some $x \in K$. By P3, $Rac*x*$. □

PROPOSITION 6. *BpconR (BsconR) is complete in respect of BpconR models (BsconR models).*

PROOF. It is clear that we just have to prove that p1-p3 hold canonically. Well, it is accomplished similarly as in Proposition 4 (in fact, it suffices to prove that p1 (or p4) and p2 (or p3) hold canonically. See the proof of proposition 3). □

We finish this paper with a note. We conjectured in [2] that full reductio could not be introduced in Bmr (see Introduction) because of the absence of prefixing and suffixing in B+. We also showed how to introduce full reductio adding prefixing to B+. Interestingly enough, as discussed in §5, this seems to be exactly the case in the logic Bconr. But in Bconr negation is not constructive, it is not defined with a falsity constant (but with a negative connective), it is semantically explained with the "Routley star" (not intuitionistically modelled) and finally, Bconr is considerably stronger than Bmr, which, in fact, is strictly included in it.

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