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A class of implicative expansions of Kleene's strong logic, a subclass of which is shown functionally complete via the precompleteness of Łukasiewicz's 3-valued logic Ł3

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Abstract The present paper is a sequel to [16]. A class of implicative expansions of Kleene's 3-valued logic functionally including Łukasiewicz's logic Ł3 is defined. Several properties of this class and/or some of its subclasses are investigated. Properties contemplated include functional completeness for the 3-element set of truth-values, presence of natural conditionals, variable-sharing property (vsp) and vsp-related properties.

Keywords 3-valued logic · Kleene's strong 3-valued logic · Łukasiewicz's 3-valued logic · functional completeness · variable-sharing property · natural conditionals.

1 Introduction

The present paper is a sequel to [16]. We define a class of implicative expansions of Kleene's strong 3-valued logic functionally including Łukasiewicz's logic Ł3 (cf. [10]). Several properties of this class and/or some of its subclasses are examined. One of the properties taken into account is functional completeness

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for the 3-element set of truth-values. A detailed record of the main results in the paper is provided at the end of §2.

Kleene's strong 3-valued matrix MK3 was introduced in [8] in the context of the treatment of partial recursive functions. The matrix MK3 (our label) can be defined as shown in Definition 2.4 below. The connectives in the propositional language are conjunction, disjunction and negation. We can take either 2 as the only designated value or else both 1 and 2. In the former case, 1 can be interpreted as neither truth nor falsity; in the latter, as both truth and falsity. The value 2 is, of course, truth while 0 is falsity.

There are many possibilities for expanding the matrix MK3 with a conditional function. In what follows, let us focus on Tomova's notion of a natural conditional (cf. [20]) and the modification of this notion we proposed in [16].

Traditionally, the conditional (or "implication" as it is named in [21] —cf. pp. 277, ff.) is required to meet the following condition (c) when interpreted in a matrix semantics: (c) $a \to b = t$ iff $a \leq b$, where t is the greatest element in the set of logical values, no other designated elements being considered in this set (cf., e.g., [21], p. 277; or [13], p. 179, ff.). But property (c) is only predicable of strong logics. For instance, a logic needs to have the rule "verum e quodlibet", "If A, then $B \to A$", in order to fulfill (c) (cf. [21], pp. 227-228; "verum e quodlibet" means "A true proposition follows from any proposition"). In this sense, Tomova's notion of a natural conditional, introduced in [20] can be viewed as an attempt to extend the set of implications worthy of the name beyond the restricted limits imposed by condition (c).

Tomova's notion can be defined as follows (cf. Definition 2.13 below). Given a matrix semantics, a conditional is *natural* if requirements (1)-(3) are fulfilled. (1) It coincides with the classical conditional when restricted to the classical values T and F (2 and 0, respectively, in the notation of this paper —cf. Definitions 2.4 and 2.13); (2) it satisfies *Modus Ponens*; (3) it is assigned a designated value whenever the value assigned to its antecedent is less than, or equal to the value assigned to the consequent.

There are very interesting implicative expansions of MK3 satisfying Tomova's definition which do not have an implication in the traditional sense though. To take only a couple of examples, the paraconsistent logic Pac (cf. [7] and references therein) or the quasi-relevant logic RM3 (cf. [1], [3]) are logics having natural conditionals (cf. [20], [16] for more very interesting logics having natural conditionals). Nevertheless, it has to be remarked that an important family of logics, relevant logics, is not covered by Tomova's definition. Mainly because of this fact, but also taking into consideration other reasons, we have weakened the condition (3) in the definition of a natural conditional to (3′): a conditional is assigned a designated value whenever its antecedent and its consequent are assigned the same value. In [16], we have delimited the set of all implicative expansions of MK3 having natural conditionals in the extended sense defined by us, as well as its subsets formed by logics functionally including Łukasiewicz's 3-valued logic Ł3. In addition, we have shown that some of the new introduced logics are relevant logics in the sense that they have the variable-sharing property (vsp) and/or other vsp-related properties (below, in Remark 2.12, the main results in [16] are recorded).

However, there are interesting logics not yet covered by the extended definition introduced by us. A conspicuous example is Priest's Logic of Paradox LP that does not satisfy Modus Ponens (cf. [12]). Consequently, in the present paper, we consider implicative expansions of MK3 that only need to fulfill the following conditions: (a) $0 \to 2 = 2$; (b) $1 \to 1 = 2$; (c) $2 \to 0 = 0$. (We recall that 2, 0 and 1 represent truth, falsity and neither truth nor falsity —or both truth and falsity—, respectively; cf. Definitions 2.4 and 2.13.) It is proved that all implicative expansions of MK3 complying with conditions (a), (b) and (c) functionally include Łukasiewicz's Ł3. Also, it is shown that some of them are functionally complete for the 3-element set of truth-values. Finally, it is investigated which of the new implicative expansions of MK3 introduced have one or more of the properties (1), (2) and (3′) defining natural conditionals (cf. Definition 2.13), as well as the vsp and vsp-related properties

The paper is organized as follows (let us provisionally refer by T1 —cf. Definition 2.6 below— to the set of all implicative expansions of MK3 fulfilling conditions (a), (b) and (c) stated above). In §2, we record some preliminary definitions as used in the paper. Also, we recall the main results obtained in [16] and set the aims of the present paper at the end of the section. In §3, we display some unary and binary connectives definable in each member in T1. In §4, we prove that Ł3 is definable from each implicative expansion of MK3 in T1. Also, we delimit the elements in T1 definable from Ł3. Finally, we investigate other 3-valued expansions of MK3 definable from T1 and/or Ł3. In §5, we define the subset of T1 being functionally complete for the 3-element set of truth-values. Next, we investigate how T1 behaves w.r.t. the properties defining a natural conditional in §6, and w.r.t. the vsp and vsp-related properties in §7. Finally, in §8, the paper is ended with some concluding remarks.

The present paper is a sequel to [16]. We use some of the results obtained there, but the following pages can be read independently from [16].

2 Preliminary notions and results

In this section, we record some preliminary notions as used in the present paper (of course, there are alternative definitions of these notions).

Definition 2.1 (Language) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.

Definition 2.2 (Logical matrix) A (logical) matrix is a structure $(\mathcal{V}, D, \texttt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); and (3) $\texttt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \texttt{F}$ such that $\mathcal{V}^n \to \mathcal{V}$. An M-interpretation is a function from the set of all wffs to $\mathcal{V}$ according to the functions in $\texttt{F}$.

In this paper logics are *prima facie* considered as structures determined by matrices. In particular, logics are defined as follows.

Definition 2.3 (Logics) Given a matrix M, a logic LM is a structure ($\mathcal{L}$, $\vDash_{\mathrm{M}}$) where $\mathcal{L}$ is a propositional language and $\vDash_{\mathrm{M}}$ is a (consequence) relation defined on $\mathcal{L}$ according to M as follows: for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I ($I(\Gamma) \in D$ iff $I(A) \in D$ for all $A \in \Gamma$). In particular, $\vDash_{\mathrm{M}} A$ (A is M-valid) iff $I(A) \in D$ for all M-interpretations I.

Next, Kleene's strong 3-valued matrix is defined. Notice that we can choose only 2 or else 1 and 2 as designated values. (Kleene uses $\mathfrak{t}$, $\mathfrak{f}$ and $\mathfrak{u}$ instead of 2, 0 and 1, respectively. The latter have been chosen in order to use the tester in [6], in case the reader needs one. Also, to put in connection the results in the present paper with previous and subsequent work by us —cf. the concluding remarks in section 8.)

Definition 2.4 (Kleene's strong 3-valued matrix) The propositional language consists of the connectives $\wedge, \vee, \neg$. Kleene's strong 3-valued matrix, MK3 (our label), is the structure ($\mathcal{V}, D, \texttt{F}$) where (1) $\mathcal{V} = \{0, 1, 2\}$ and it is ordered as shown in the following lattice:

(2) $D = \{1, 2\}$ or $D = \{2\}$; (3) $\texttt{F} = \{f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\neg$ is an involution with $f_\neg(2) = 0, f_\neg(0) = 2$ and $f_\neg(1) = 1$. We display the tables for $\wedge$, $\vee$ and $\neg$:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\neg$	0
0	2
1	1
2	0

The notion of an MK3-interpretation is defined according to the general Definition 2.2.

The logic determined by MK3 can be named here $\mathrm{K3}^1$ (only one designated value) or $\mathrm{K3}^2$ (two designated values) (cf. [4], §2.4 on these logics). Then, all logics defined in this paper are expansions of either $\mathrm{K3}^1$ or else $\mathrm{K3}^2$.

Next, the notion of an implicative expansion of MK3 is defined and the set of implicative expansions of MK3 primarily investigated in the present paper is displayed.

Definition 2.5 (Implicative expansions of MK3) Let M be an expansion of MK3 built up by adding any $f_{\rightarrow}$-function. It is said that M is an implicative expansion of MK3.

Definition 2.6 (The general table T1) The general table T1 (a_i $(1 \leq i \leq 6) \in \{0, 1, 2\}$) is composed of the following 729 $f_{\rightarrow}$-functions:

T1

$\rightarrow$	0	1	2
0	a_1	a_2	2
1	a_3	2	a_4
2	0	a_5	a_6

In the present paper, we are primarily, but not exclusively, concerned with the implicative expansions of MK3 that can be built up by adding to it any of the $f_{\rightarrow}$-functions included in T1. We note that addition of any of the $f_{\rightarrow}$-functions in T1 to MK3 results in an implicative expansion of MK3. Nevertheless, other general tables include, for the sake of generality, besides not definable binary functions in MK3, other that are definable. This is, for instance, the case of T5 and T6, displayed below.

In subsequent sections, we investigate T1. Meanwhile, more definitions and referential conventions useful in the paper are introduced.

In many cases, a set of implicative expansions of MK3 (or a set of binary 3-valued functions in general) is defined in the pages to follow by means of a general table such as T1, for instance. Thus, we introduce the following definition.

Definition 2.7 (T-logics, t-logics) Let Tn be a general table and tm be one of the functions in Tn. By a Tn-logic (resp., tm-logic), we mean the logic LM, M being an expansion of MK3 built from a function in table Tn (resp., the function tm in Tn).

Next the notions of "propositional connectives definable in a logic", "functionally equivalent logics" and "functional completeness" are defined.

Definition 2.8 (Connectives definable in a logic) Let L be a logic determined by the matrix M (cf. Definition 2.3). An n-ary connective c is definable in L iff there is a formula A in L in which only connectives of L occur and at most the propositional variables $p_1, ..., p_n$ and such that for any M-interpretation I, $I(c(p_1, ..., p_n)) = I(A)$.

Definition 2.9 (Functional inclusion) The logic L is functionally included in the logic L$'$ iff every primitive connective of L is definable in L$'$.

Definition 2.10 (Functional equivalence) The logics L and L$'$ are functionally equivalent iff L is functionally included in L$'$ and L$'$ is functionally included in L.

Definition 2.11 (Functional completeness) Let $\mathcal{V}$ be a set of n elements (truth-values). A logic L is functionally complete for $\mathcal{V}$ if every m-ary ($1 \leq m \leq n$) connective definable in $\mathcal{V}$ is definable in L.

Now, let us recall the main results in [16] we are going to extend in the following pages. Then, we can state the main results in the present paper.

Remark 2.12 (Previous results) Consider the subsequent general implicative tables TI, TII, TIII and TIV (a_i ($1 \leq i \leq 3$) $\in \{0, 1, 2\}$; $b_1 \in \{1, 2\}$; $c_1 \in \{0, 1\}$; designated values are starred).

TI

$\to$	0	1	2
0	2	a_1	2
*1	0	b_1	a_2
*2	0	a_3	2

TII

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
*2	0	c_1	2

Table I can be subdivided in the general tables TIII and TIV

TIII

$\to$	0	1	2
0	2	a_1	2
*1	0	2	a_2
*2	0	a_3	2

TIV

$\to$	0	1	2
0	2	a_1	2
*1	0	1	a_2
*2	0	a_3	2

In [16], facts (1), (2) and (3) remarked below are proved.

1. The 54 TII-logics are functionally equivalent to Łukasiewicz's 3-valued logic Ł3.
2. The 27 TIII-logics are functionally equivalent to the paraconsistent logic J3 and, consequently, they are also functionally equivalent to Ł3.
3. The 27 TIV-logics are functionally included in (but do not include) any of the TIII-logics. Moreover, the 27 TIV-logics are not in general functionally equivalent to each other.

We note a remark on the implicative extensions of MK3 definable by means of (the $f_\to$-functions) in TI and TII. Consider the following definition of a "natural conditional", an extension of the notion introduced by Tomova in [20].

Definition 2.13 (Natural conditionals) Let $\mathcal{V}$ and D be defined as in Definition 2.4. Then, an $f_\to$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:

1. $f_\to$ coincides with (the $f_\to$-function for) the classical conditional when restricted to the subset $\{0, 2\}$ of $\mathcal{V}$.
2. $f_\to$ satisfies Modus Ponens, that is, for any $a, b \in \mathcal{V}$, if $f_\to(a, b) \in D$ and $a \in D$, then $b \in D$.
3. For any $a, b \in \mathcal{V}$, $f_\to(a, b) \in D$ if $a = b$.

Then, notice that all implicative expansions of MK3 definable by using TI and TII do exhibit natural conditionals, which is not (by far) the case with the implicative expansions definable by using T1. Nevertheless, we refer the reader to sections 6 and 7, where some sets of very interesting implicative expansions of MK3 contained in T1 are briefly discussed.

Now, we can state the main results in the paper.

1. The 729 T1-logics functionally include Łukasiewicz's 3-valued logic Ł3.
2. The 243 T2-logics and 162 T3-logics, included in the set of all T1-logics, and contained in the general tables T2 and T3 displayed below (a_i $(1 \leq i \leq 5) \in \{0, 1, 2\}$; $d_1 \in \{0, 2\}$) are the only T1-logics functionally complete for $\mathcal{V} = \{0, 1, 2\}$ (cf. Definition 2.11).

T2

$\rightarrow$	0	1	2
0	1	a_1	2
1	a_2	2	a_3
2	0	a_4	a_5

T3

$\rightarrow$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	1

3. The only T1-logics that Ł3 functionally includes are displayed in the following general table T4 (a_i $(1 \leq i \leq 4) \in \{0, 1, 2\}$; d_i $(1 \leq i \leq 2) \in \{0, 2\}$).

T4

$\rightarrow$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	d_2

4. In addition to the T1-logics which have been mentioned above in each case, all the T5-logics (resp., T6-logics) definable from the general tables T5 and T6 (a_i $(1 \leq i \leq 7) \in \{0, 1, 2\}$; d_i $(1 \leq i \leq 4) \in \{0, 2\}$) are, in their turn, definable from Ł3 (resp., each T1-logic).

T5

	0	1	2
0	d_1	a_1	d_2
1	a_2	a_3	a_4
2	d_3	a_5	d_4

T6

	0	1	2
0	a_1	a_2	d_1
1	a_3	a_4	a_5
2	d_2	a_6	a_7

5. Finally, in sections 6 and 7, it is discussed which subsets of the implicative expansions of MK3 included in T1 have one or more of the properties defining a natural conditional (cf. Definition 2.13), as well as some other interesting properties such as the "variable-sharing property" (vsp) and vsp-related properties such as the "quasi variable-sharing property" (qvsp) and the "weak relevant property" (wrp).

3 Some connectives definable from T1

In order to prove the results (1)-(5) described at the end of preceding section, we need to prove some preliminary facts. In particular, we remark some unary and binary connectives definable from the general table T1. We begin by stating some useful definitional conventions frequently employed throughout the paper.

Criterion 3.1 (Sets built up from $\mathcal{V} = \{0, 1, 2\}$) *In the general tables displayed in the paper, sets of truth-values in $\mathcal{V} = \{0, 1, 2\}$ are represented similarly as in T1 and TI-TIV above. That is, a_i $(1 \leq i \leq 9) \in \{0, 1, 2\}$; b_i $(1 \leq i \leq 9) \in \{1, 2\}$; c_i $(1 \leq i \leq 9) \in \{0, 1\}$ and d_i $(1 \leq i \leq 9) \in \{0, 2\}$.*

Criterion 3.2 (Proof of functional equivalence) *Let M and M′ be two expansions of MK3 and LM and LM′ be the logics determined by them (cf. Definitions 2.3 and 2.5). We show that LM is functionally included in LM′ by proving that the new functions added to MK3 in order to build up M are definable in M′. (An n-ary function $f*$ is definable in $M = (\mathcal{V}, D, \mathtt{F})$ iff there are fp and $e_1, .., .e_m$ such that $fp \in \mathtt{F}$ and either $e_i \in \mathcal{V}$ or else $e_i \in \mathtt{F}$ $(1 \leq i \leq m)$, and for all $a_1, ..., a_n \in \mathcal{V}$, $f * (a_1, ..., a_n) = fp(e_i, ..., e_m)$.)*

Consequently, the proofs of definability of some table by another given in the following pages have to be understood in this sense and as given in the context of two or more expansions of MK3.

Criterion 3.3 (Prop. of a gen. table; definable tables and connectives) *Let Tn be a general table as, say, T1 or TII. For instance, we say that TII is included in (or definable from) T1 meaning that each expansion of MK3 built by adding any function in TII is functionally included in each expansion of MK3 built by adding any function in T1. The same manner of speaking can be used in the case of a general table and a particular one, say, the characteristic implicative table tL3 of L3. Also, we can say, for instance, that a given connective c is "definable from" TII, meaning that c is definable in each expansion of MK3 built up by adding any of the functions in TII. Finally, sometimes we say that P is a property of, say, T1, meaning that it is a property of each expansion of MK3 definable from T1, as the latter notion has been defined above.*

In what follows, some unary connectives are defined.

Lemma 3.4 (An inconsistency connective) *The inconsistency connective $\bullet$, given by the truth-table*

	$\bullet$
0	0
1	1
2	0

is definable from any expansion of MK3.

Proof We set $\bullet A = A \wedge \neg A$, for any wff A.

Lemma 3.5 (Additional negation connectives) *The additional negation connectives* $\overset{\bullet}{\neg}$ *and* $\overset{\circ}{\neg}$ *given by the truth-tables*

	$\overset{\bullet}{\neg}$
0	2
1	2
2	0

	$\overset{\circ}{\neg}$
0	2
1	0
2	0

are definable from T1.

Proof We set $\overset{\bullet}{\neg}A = A \to \neg A$ and $\overset{\circ}{\neg}A = \neg(\neg A \to A)$, for any wff A.

Lemma 3.6 (Truth and falsity conversing connectives) *The truth conversing connective* $\overset{\bullet}{t}$ *and the false conversing connective* $\overset{\bullet}{f}$*, given by the truth-tables*

	$\overset{\bullet}{t}$
0	2
1	2
2	2

	$\overset{\bullet}{f}$
0	0
1	0
2	0

are definable from T1.

Proof We set $\overset{\bullet}{t}A = \overset{\bullet}{\neg} \bullet A$ and $\overset{\bullet}{f}A = \neg\overset{\bullet}{t}A$ for any wff A.

Next, several binary connectives are defined, but firstly, a couple more of conventions are introduced.

Criterion 3.7 (Connectives and tables I) *Let tn be a truth-table. By kn, we refer to the connective defined by it. Then, given the binary connectives ki and kj and additional connective* $\overset{m}{\to}$ *(defined by table tm), by* $ti \cup tj$ *(resp.,* $ti \cap tj$*,* $ti \overset{m}{\supset} tj$*) we refer, for any wffs* A, B*, to the wff* $(A\ ki\ B) \vee (A\ kj\ B)$ *(resp.,* $(A\ ki\ B) \wedge (A\ kj\ B)$*,* $(A\ ki\ B) \overset{m}{\to} (A\ kj\ B)$*). A similar convention is used in the case of general tables in their relation to either other general tables or else to a particular table defining some connective.*

Example 3.8 (Connectives and tables II) We have (cf. Criterion 3.1):

(a) $TI \cap TII =$

$\to$	0	1	2
0	2	a_1	2
1	0	b_1	a_2
2	0	c_1	2

(b) $TI \cup TII =$

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	a_4	2

(c) $TI \overset{RM3}{\supset} TII =$

$\to$	0	1	2
0	2	a_1	2
1	2	2	a_2
2	2	a_3	2

$\overset{RM3}{\supset}$ *is given by the characteristic $f_\to$-table of the quasi-relevant logic RM3 (cf. [1], [3]), i.e.,*

$\to$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

Criterion 3.9 (Connectives and tables III) *Given general tables Ti, Tj, we note that $Ti \cup Tj$ is not the set of all tables contained in Ti or Tj. We use the symbol $\dot{\cup}$ to refer to the aforementioned set. Thus, for example, as pointed out in Example 3.8,*

$TI \cup TII =$

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	a_4	2

$TI \dot{\cup} TII$, however, is the set of tables

$\to$	0	1	2
0	2	a_1	2
1	a_2	b_1	a_3
2	0	a_4	2

(Cf. Criterion 3.1).

Lemma 3.10 (Some additional binary connectives) *The additional binary connectives $k1$ through $k12$, given by the truth-tables t1 through t12, respectively,*

t1	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

t2	0	1	2
0	2	0	0
1	0	0	0
2	0	0	0

t3	0	1	2
0	2	0	0
1	0	0	0
2	0	1	2

t4	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

t5	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

t6	0	1	2
0	2	2	0
1	2	2	1
2	2	2	2

t7	0	1	2
0	2	2	2
1	2	2	2
2	2	2	0

t8	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

t9	0	1	2
0	2	0	0
1	2	1	1
2	2	2	2

t10	0	1	2
0	2	2	2
1	2	1	2
2	2	2	2

t11	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

t12	0	1	2
0	2	2	2
1	2	2	2
2	2	2	2

are definable from T1.

Proof We set, for any wffs A, B, (t1) $A\ k1\ B = (A \wedge B) \wedge \overset{\bullet}{\neg}\overset{\bullet}{\neg}A$; (t2) $(A\ k2\ B) = \overset{\circ}{\neg}A \wedge \overset{\circ}{\neg}B$; (t3) $A\ k3\ B = (A\ k1\ B) \vee (A\ k2\ B)$ (i.e., t3 = t1 ∪ t2); (t4) $A\ k4\ B = A \wedge \overset{\circ}{\neg}B$; (t5) $A\ k5\ B = \overset{\bullet}{\neg}A \vee B$; (t6) $A\ k6\ B = A \vee \overset{\bullet}{\neg}B$; (t7) $A\ k7\ B = \overset{\bullet}{\neg}A \vee \overset{\bullet}{\neg}B$; (t8) $A\ k8\ B = \overset{\circ}{\neg}A \vee B$; (t9) $A\ k9\ B = A \vee \overset{\circ}{\neg}B$; (t10) t10 = t8 ∪ t9; (t11) $(A\ k11\ B) = \neg(\overset{\circ}{\neg}A \wedge \overset{\circ}{\neg}B)$; (t12) t12 = t6 ∪ t10.

In the next section, it is proved that Ł3 is definable from T1 (we use Criterions 3.1 through 3.3 and 3.8).

4 Ł3 is definable from T1. Tables in T1 definable from tŁ3. Other 3-valued tables definable from T1 and/or tŁ3

We prove:

Theorem 4.1 (Ł3 is definable from T1) *The $f_{\rightarrow}$-table tŁ3, i.e.,*

tŁ3

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

which is characteristic of Łukasiewicz's 3-valued logic Ł3, is definable from T1, i.e.,

T1

$\rightarrow$	0	1	2
0	a_1	a_2	2
1	a_3	2	a_4
2	0	a_5	a_6

(cf. Definition 2.6).

Proof As remarked in section 2 (Remark 2.12), Ł3 is definable from the general table TII

TII

$\rightarrow$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	c_1	2

Then, table TIIa, included in TII,

TIIa

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	1	2

is definable from T1, t3 and t5 as follows:

T1 $\cup$

t3	0	1	2
0	2	0	0
1	0	0	0
2	0	1	2

=

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	b_1	2

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	b_1	2

$\cap$

t5	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

= TIIa

In what follows, we delimit the set of all implicative expansions of MK3, built upon the 729 tables in T1, functionally included in Ł3. We note a preliminary remark (we recall that tables T1-T6 are described at the end of section 2).

Remark 4.2 (3-valued functions not definable from Ł3) The 15795 functions contained in the general tables T7a, T7b, T7c and T7d

T7a

	0	1	2
0	1	a_1	a_2
1	a_3	a_4	a_5
2	a_6	a_7	a_8

T7b

	0	1	2
0	d_1	a_1	1
1	a_2	a_3	a_4
2	a_5	a_6	a_7

T7c

	0	1	2
0	d_1	a_1	d_2
1	a_2	a_3	a_4
2	1	a_5	a_6

T7d

	0	1	2
0	d_1	a_1	d_2
1	a_2	a_3	a_4
2	d_3	a_5	1

are not definable from Ł3.

The proof of this fact is immediate, since it is not possible to define from Ł3 a function taking the value 1 if it only has the value 0 and 2 as its arguments (cf. the classical paper [18]).

Now, we prove:

Theorem 4.3 (Functions in T1 definable from Ł3) *The general table T4 (cf. §1)*

T4

$\to$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	d_2

contains all functions in T1 definable from Ł3.

Proof In the first place, we prove that T4 is definable from Ł3. As it is known, 3-valued logic Ł3 is precomplete in the sense that any 3-valued C-extending function (i.e., a function which coincides with the corresponding classical function when restricted to the classical values) is definable by the functions of Ł3 (this was firstly proved in [5]; cf. also [2]). Using this fact, it is immediate that the general table T4a

T4a

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	a_4	2

is definable from Ł3. Consider now the following general and particular tables (cf. Lemma 3.10 —we note that this lemma is provable in T1 and that tŁ3 is one of the tables in T1).

T4b

$\to$	0	1	2
0	0	a_1	2
1	a_2	2	a_3
2	0	a_4	2

T4c

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	a_4	0

T4d

$\to$	0	1	2
0	0	a_1	2
1	a_2	2	a_3
2	0	a_4	0

t7	0	1	2
0	2	2	2
1	2	2	2
2	2	2	0

t11	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

We have: T4a $\cap$ t11 = T4b; T4a $\cap$ t7 = T4c; T4b $\cap$ T4c = T4d. But it is clear that T4a $\dot{\cup}$ T4b $\dot{\cup}$ T4c $\dot{\cup}$ T4d = T4.

Now, consider tables T2 and T3 (cf. §2):

T2

$\to$	0	1	2
0	1	a_1	2
1	a_2	2	a_3
2	0	a_4	a_5

T3

$\to$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	1

Clearly, T4 $\dot{\cup}$ T2 $\dot{\cup}$ T3 = T1. But T2 and T3 are not definable from tŁ3: they are included in T7a and T7d, respectively (cf. Remark 4.2). Consequently, T4 is the set of all functions in T1 definable from Ł3. (Below, it is proved that T2-logics together with T3-logics form the set of all T1-logics that are functionally complete —cf. Definition 2.7, Theorem 5.1 and Remark 5.2.)

In the sequel, we describe other binary functions definable from T1 and/or Ł3. We begin by remarking some particular binary connectives definable from Ł3.

Lemma 4.4 (Other binary connectives definable from Ł3) *The additional binary connectives* $k13$*,* $k14$ *and* $k15$*, given by the truth-tables t13, t14 and t15, respectively,*

t13	0	1	2
0	0	0	0
1	0	0	0
2	2	0	0

t14	0	1	2
0	2	2	0
1	2	2	2
2	2	2	2

t15	0	1	2
0	2	2	2
1	2	0	2
2	2	2	2

are definable from Ł3.

Proof Let $\stackrel{x}{\to}$ (resp., $\stackrel{y}{\to}$) be a f-function in T4 (cf. Theorem 4.3) such that $f_{\to}(2,1) = 0$ (resp., $f_{\to}(2,1) = 2$). We set, for any wffs A, B, $A\ k13\ B = (A\ k12\ B) \stackrel{x}{\to} (A\ k4\ B)$ (i.e., t13 = t12 $\stackrel{x}{\supset}$ t4); $A\ k14\ B = (A\ k12\ B) \stackrel{y}{\to} (A\ k6\ B)$ (i.e., t14 = t12 $\stackrel{y}{\supset}$ t6); $A\ k15\ B = (A\ k12\ B) \stackrel{x}{\to} (A\ k10\ B)$ (i.e., t15 = t12 $\stackrel{x}{\supset}$ t10).

Next, we select the set of all binary 3-valued tables definable from Ł3.

Theorem 4.5 (All binary 3-valued functions definable from Ł3) *The 3888 functions in the general table T5*

T5

	0	1	2
0	d_1	a_1	d_2
1	a_2	a_3	a_4
2	d_3	a_5	d_4

are the only binary 3-valued functions definable from Ł3.

Proof We begin by proving that T5 is definable from tŁ3. We use tables t10, t13, t14 and t15 (cf. Lemmas 3.10 and 4.3). Given the general table T4, definable from Ł3 (cf. Theorem 4.3),

T4

$\to$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	d_2

we have:

T4 $\cap$

t15	0	1	2
0	2	2	2
1	2	0	2
2	2	2	2

= T5a

	0	1	2
0	d_1	a_1	2
1	a_2	0	a_3
2	0	a_4	d_2

T4 $\cap$

t10	0	1	2
0	2	2	2
1	2	1	2
2	2	2	2

= T5b

	0	1	2
0	d_1	a_1	2
1	a_2	1	a_3
2	0	a_4	d_2

T4 $\dot{\cup}$ T5a $\dot{\cup}$ T5b = T5c

	0	1	2
0	d_1	a_1	2
1	a_2	a_3	a_4
2	0	a_5	d_2

Next, we prove:

T5c $\cup$

t13	0	1	2
0	0	0	0
1	0	0	0
2	2	0	0

= T5d

	0	1	2
0	d_1	a_1	2
1	a_2	a_3	a_4
2	2	a_5	d_2

T5c $\cap$

t14	0	1	2
0	2	2	0
1	2	2	2
2	2	2	2

= T5e

	0	1	2
0	d_1	a_1	0
1	a_2	a_3	a_4
2	0	a_5	d_2

T5e $\cup$

t13	0	1	2
0	0	0	0
1	0	0	0
2	2	0	0

= T5f

	0	1	2
0	d_1	a_1	0
1	a_2	a_3	a_4
2	2	a_5	d_2

But it is clear that t5c $\dot{\cup}$ t5d $\dot{\cup}$ t5e $\dot{\cup}$ t5f = T5.

On the other hand, it follows from Remark 4.2 together with the proof just given that T5 contains all binary 3-valued functions definable from Ł3.

Concerning other 3-valued functions definable from T1, we have:

Proposition 4.6 (Other 3-valued functions definable from T1) *The 8748 binary functions in the general table T6*

T6

	0	1	2
0	a_1	a_2	d_1
1	a_3	a_4	a_5
2	d_2	a_6	a_7

are definable from T1.

Proof Immediate, given Theorem 4.4: T5 $\dot{\cup}$ T1 = T6.

5 Functional completeness in T1

As pointed out above, T4 $\dot{\cup}$ T2 $\dot{\cup}$ T3 = T1 (cf. the proof of the Theorem 4.3). Now, the 324 implicative expansions of MK3 that can be built from the functions in T4 are not functionally complete: T4 is included in T5 (cf. Theorem 4.4), but T5 is not functionally complete (cf. Criterion 3.3, Remark 4.2). Nevertheless, the situation is exactly the reverse in the case of T2 and T3.

Theorem 5.1 (T2 and T3 are functionally complete) *Tables T2 and T3 are functionally complete (cf. Criterion 3.3). That is, the 405 implicative expansions of MK3 that can be built up from the functions in T2 and T3*

T2

$\to$	0	1	2
0	1	a_1	2
1	a_2	2	a_3
2	0	a_4	a_5

T3

$\to$	0	1	2
0	d_1	a_1	2
1	a_2	2	a_3
2	0	a_4	1

are functionally complete.

Proof In his classical paper [18], Słupecki shows that the addition of the unary connective T, given by the table

	T
0	1
1	1
2	1

to Ł3 results in a logic functionally complete for $\mathcal{V} = \{0, 1, 2\}$ (cf. Definition 2.11 —Słupecki uses 1 and 2 instead of 2 and 1, respectively). Now, above it has been proved that Ł3 is definable from T2 and from T3, since it is definable from T1 (Theorem 4.1) and T2 and T3 are definable from T1 (cf. Proposition 4.6). Consequently, in order to prove that the 405 implicative expansions of MK3 definable from the functions in T2 and T3 are functionally complete, it suffices to prove that the unary connective T can be defined in each one of them. With a view to fulfill this task, we use (a) the inconsistency operator $\bullet$, whose table is

	$\bullet$
0	0
1	1
2	0

(cf. Lemma 3.4); and (b) the general tables $T2_1$ and $T3_1$ resulting from changing in T2 and T3 $f_{\to}(1, 1) = 2$ for $f_{\to}(1, 1) = 1$

$T2_1$

$\to$	0	1	2
0	1	a_1	2
1	a_2	1	a_3
2	0	a_4	a_5

$T3_1$

$\to$	0	1	2
0	d_1	a_1	2
1	a_2	1	a_3
2	0	a_4	1

(We note that $T2_1$ and $T3_1$ are definable from T2 and T3, respectively, since $T2_1$ and $T3_1$ are included in T6 (cf. Proposition 4.6), in its turn, definable from T1. Actually, we have T2 $\cap$ t10 (resp. T3 $\cap$ t10) = $T2_1$ (resp. $T3_1$).) Then, we set, for any wff A, $TA = \bullet A \to \bullet A$ in the case of $T2_1$, and $TA = \neg \bullet A \to \neg \bullet A$, in the case of $T3_1$.

We note the following remark.

Remark 5.2 (Functional completeness in T1) We remark that the general tables T2 and T3 contain all functionally complete implicative expansions of MK3 that can be built from T1: T2 $\dot{\cup}$ T3 $\dot{\cup}$ T4 = T1, but T4 is not functionally complete (cf. Theorem 4.5).

6 Natural conditionals

We examine which functions in T1 and $T1_1$ fulfill one or more of the conditions defining a natural conditional (cf. Definition 2.13). The general table $T1_1$ is the result of changing in T1 $f_{\to}(1,1) = 2$ for $f_{\to}(1,1) = 1$ (notice that it follows from Proposition 4.6 that $T1_1$ is definable from T1).

$T1_1$

$\to$	0	1	2
0	a_1	a_2	2
1	a_3	1	a_4
2	0	a_5	a_6

We have the following propositions, whose proof is immediate by inspection.

Proposition 6.1 (Functions in T1 and $T1_1$ verifying MP) *(a) Two designated values. The 243 functions in T8 and the 243 functions in T9*

T8

$\to$	0	1	2
0	a_1	a_2	2
1	0	2	a_3
2	0	a_4	a_5

T9

$\to$	0	1	2
0	a_1	a_2	2
1	0	1	a_3
2	0	a_4	a_5

are the only functions in T1 and $T1_1$ verifying Modus Ponens, respectively.

(b) Only one designated value. The 486 functions in T10 and the 486 functions in T11

T10

$\to$	0	1	2
0	a_1	a_2	2
1	a_3	2	a_4
2	0	c_1	a_5

T11

$\to$	0	1	2
0	a_1	a_2	2
1	a_3	1	a_4
2	0	c_1	a_5

are the only functions in T1 and $T1_1$ verifying Modus Ponens, respectively.

Proposition 6.2 (C-extending functions in T1 and $T1_1$) *The 162 functions in T12*

T12

$\to$	0	1	2
0	2	a_1	2
1	a_2	b_1	a_3
2	0	a_4	2

are the only C-extending $f_{\to}$-functions in T1 and $T1_1$ (a C-extending function is a function which coincides with the corresponding classical function when restricted to the classical values).

Proposition 6.3 ($a \to b \in D$ iff $a = b$ in T1 and $T1_1$) *Condition (3) in Definition 2.13 is the following: $a \to b \in D$ iff $a = b$. (a) Two designated values. The 648 functions in T13:*

T13

$\to$	0	1	2
0	b_1	a_1	2
1	a_2	b_2	a_3
2	0	a_4	b_3

are the only functions in T1 and T1$_1$ complying with condition (3) in Definition 2.13.

(b) Only one designated value: The 81 functions in T14

T14

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	a_4	2

are the only functions in T1 fulfilling condition (3) in Definition 2.13 (there are no functions in T1$_1$ fulfilling condition (3) when 2 *is the only designated value).*

Remark 6.4 (C-extending functions with condition (3)) We note that the 162 functions in T12 are the only C-extending functions fulfilling condition (3) (Definition 2.13) in tables T1 and T1$_1$ when 1 and 2 are designated values. On the other hand, if 2 is the only designated value, the 81 functions in T14 are the only C-extending functions fulfilling condition (3) (Definition 2.13). Interesting implicative expansions of MK3 not verifying MP can be found in T12, such as, for example, Priest's Logic of Paradox LP (cf. [12]).

Remark 6.5 (Natural conditionals) The only natural conditionals (according to Definition 2.13) in T1 and T1$_1$ are those in the general tables TI and TII studied in [15] and recalled in Remark 2.12.

7 Variable-sharing property and related properties

We explore the behavior of the implicative expansions of MK3 that can be built upon the tables in T1 and T1$_1$ w.r.t. the variable-sharing property (vsp) and related properties. The vsp and these properties can be defined as follows (cf. [16] and references therein).

Definition 7.1 (Variable-sharing property —vsp) Let L be a logic defined upon the matrix M. L has the "variable-sharing property" (vsp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable.

Definition 7.2 (Quasi variable-sharing property —qvsp) Let L be a logic defined upon the matrix M. L has the "quasi variable-sharing property" (qvsp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable or else both A and B are M-valid.

Definition 7.3 (Weak relevant property —wrp) Let L be a logic defined upon the matrix M. L has the "weak relevant property" (wrp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable or else both $\neg A$ and B are M-valid.

In the sequel, we prove some facts regarding these properties (cf. Definition 2.7).

Proposition 7.4 (T1-logics, the vsp and qvsp) *Let L be a T1-logic. Then, L lacks the vsp and the qvsp.*

Proof We use the truth-conversing connective $\overset{\bullet}{t}$ and the falsity-conversing connective $\overset{\bullet}{f}$ (cf. Lemma 3.6). For any distinct propositional variables p, q, the wff $\overset{\bullet}{f}p \to \overset{\bullet}{t}q$ is M-valid in any implicative expansion of MK3, M, built upon any of the 729 tables in T1. But $\overset{\bullet}{f}p$ is not M-valid.

Proposition 7.5 (T1-logics and the wrp I) *Let L be a T1-logic built upon a table in T1 such that $f_{\to}(2,2) = 2$ or $f_{\to}(0,0) = 2$. Then, L lacks the wrp.*

Proof We use again the connectives $\overset{\bullet}{t}$ and $\overset{\bullet}{f}$. For any propositional variables p, q, the wffs $\overset{\bullet}{t}p \to \overset{\bullet}{t}q$ and $\overset{\bullet}{f}p \to \overset{\bullet}{f}q$ are M-valid, M being the implicative expansion of MK3 determining L. But neither $\neg\overset{\bullet}{t}p$ nor $\overset{\bullet}{f}q$ are M-valid.

We note the following remark.

Remark 7.6 (On the proofs of Propositions 7.4 and 7.5) Notice that the proofs of Propositions 7.4 and 7.5 hold, no matter if both 1 and 2, or only 2, are designated values.

Consider now the ensuing general tables T15a and T15b.

T15a

$\to$	0	1	2
0	0	a_1	2
1	a_2	2	a_3
2	0	a_4	0

T15b

$\to$	0	1	2
0	c_1	a_1	2
1	a_2	2	a_3
2	0	a_4	c_2

Given Proposition 7.5, concerning T1-logics and wrp, it remains to investigate which T1-logics contained in T15a (1 and 2 are designated values) and T15b (1 is the only designated value) have the wrp. In order to fulfill this aim we begin by noting the following remark.

Remark 7.7 (The T-operator in T15b) Let tm be a table in T15b such that $f_{\to}(0,0) = 1$ or $f_{\to}(2,2) = 1$, and let L be a T1-logic built upon tm. Similarly as in §5 (cf. Theorem 5.1), the T-operator is definable from tm, whence we can use the wff $Tp \to Tq$ (p and q are distinct variables) to show that L lacks the wrp since $\neg Tp$ (and Tq) are not valid in the expansion of MK3 built upon tm.

It is a consequence of this remark that we have to survey only the tables in T15a, both when 1 and 2 are designated values, and when 2 is the only designated one. In what follows, we examine which T15a logics have the wrp. Firstly, we contemplate the case where 1 and 2 are designated.

Proposition 7.8 (T1-logics and wrp II) *Let L be a T15a-logic,* 1 *and* 2 *being designated values. Then, L has the wrp.*

Proof Let M be the implicative expansion of MK3 determining the logic L. Suppose that there are wffs A and B not having propositional variables in common and such that $A \to B$ is M-valid but either $\neg A$ or B is not.

(a1) $\neg A$ *is not M-valid.* Then, there is an M-interpretation I such that $I(\neg A) = 0$. So, $I(A) = 2$ and $I(B) = 1$ (if $I(B) = 0$ or $I(B) = 2$, $I(A \to B) = 0$). Let I' be an M-interpretation exactly as I, except that for each propositional variable p in B, $I'(p) = 0$. Clearly $I'(B) = 0$ or $I'(B) = 2$ (since $\{0, 2\}$ is closed under $\to, \wedge, \vee$ and $\neg$) and $I'(A) = 2$ (since A and B do not share propositional variables). Then, $I'(A \to B) = 0$, contradicting the hypothesis.

(a2) B *is not M-valid.* Then, there is an M-interpretation I such that $I(B) = 0$ and $I(A) = 1$ (if $I(A) = 0$ or $I(A) = 2$, $I(A \to B) = 0$). Let now I' be an M-interpretation exactly as I except that for each propositional variable p in A $I'(p) = 0$. Similarly as in case (a1), $I'(A) = 0$ or $I'(A) = 2$ and $I'(B) = 0$, whence $I'(A \to B) = 0$, contradicting the hypothesis.

With respect to the case of only one designated value, we state the following remark.

Remark 7.9 (T15a-logics without the wrp —only 2 is designated) Let tm be a table in T15a (only 2 is designated) such that $f_{\to}(0, 1) = 2$ or $f_{\to}(1, 2) = 2$ and let L be a T1-logic built upon tm. Then, L lacks the wrp, since for distinct propositional variables p, q, $\overset{\bullet}{f}p \to (q \vee \neg q)$ and $\bullet p \to tq$ are valid in the expansion of MK3 built upon tm (cf. Lemmas 3.4 and 3.6 w.r.t. the operators $\overset{\bullet}{f}$ and $\bullet$).

In view of Remark 7.9, we are left with the following general table T15c

T15c

$\to$	0	1	2
0	0	c_1	2
1	a_1	2	c_2
2	0	a_2	0

w.r.t. the question whether there are T1-logics (with only 2 as designated value) enjoying the wrp. Regarding this question, we have the following proposition.

Proposition 7.10 (T15a-logics with the wrp —only 2 is designated) *Consider the following general tables T15d, T15e and T15f included in T15c.*

T15d

$\to$	0	1	2
0	0	c_1	2
1	c_2	2	c_3
2	0	c_4	0

T15e

$\to$	0	1	2
0	0	c_1	2
1	c_2	2	c_3
2	0	a_1	0

T15f

$\to$	0	1	2
0	0	c_1	2
1	2	2	c_2
2	0	2	0

Let L be a T1-logic built upon a table contained in one of these general tables. Then, L has the wrp.

Proof The proof follows a similar strategy as in Proposition 7.8. Actually, the cases when $\neg A$ or B are assigned 0 are proved practically in the same way. So, let us view the cases when A or B are assigned the value 1. (a) T15d. It is immediate that we can define an M-interpretation I such that $I(B) \in \{0, 2\}$ (resp., $I(A) \in \{0, 2\}$), whence $I(A \to B) \in \{0, 1\}$, contradicting the hypothesis. (b) T15e. The proof is similar to that of case c, which we next proceed to display. (c) T15f. Suppose that A is assigned the value 1, i.e., $I(A) = 1$ (cf. Proposition 7.8). Then, we can suppose $I(B) \neq 2$: otherwise $I(A \to B) \in \{0, 1\}$. So, (ci) $I(B) = 0$ or (cii) $I(B) = 1$. Subcase (ci): Define an M-interpretation I' similarly as in case (a2) in Proposition 7.8. Then, $I'(A \to B) = 0$, contradicting the hypothesis. Subcase (cii): Similarly as in case (a1) in Proposition 7.8, define an M-interpretation I' such that $I'(B) \in \{0, 2\}$ and $I'(A) = 1$. If $I'(B) = 2$, then $I'(A \to B) \in \{0, 1\}$, contradicting the hypothesis. Suppose then $I'(B) = 0$. In that case, define an M-interpretation I'' exactly as I', except that for each propositional variable p in A, $I''(p) = 0$. Then, $I''(A) \in \{0, 2\}$, $I''(B) = 0$, and consequently, $I''(A \to B) = 0$, contradicting the hypothesis.

Since no T1-logic has the vsp or the qvsp, while the wrp is only predicable of 81 T1-logics (1, 2 are designated) and 36 T1-logics (only 2 is designated), let us now explore how the situation is w.r.t. the vsp in T1_1, the result of changing in T1 $f_\to(1, 1) = 2$ for $f_\to(1, 1) = 1$.

We restrict our attention to the case of two designated values since when 2 is the only designated value, T1_1 has no interest whatsoever from the point of view we are adopting: all conditionals are no valid.

As a first step, we remark which T1_1-logics lack the vsp.

Proposition 7.11 (T1_1-logics lacking the vsp) *Let L be an implicative expansion of MK3 built upon any of the 603 functions in the tables T16 through T27. Then, L lacks the vsp.*

T16

$\to$	0	1	2
0	a_1	b_1	2
1	a_2	1	b_2
2	0	a_3	a_4

T17

$\to$	0	1	2
0	0	0	2
1	a_1	1	b_1
2	0	b_2	2

T18

$\to$	0	1	2
0	2	0	2
1	a_1	1	b_1
2	0	b_2	2

T19

$\to$	0	1	2
0	b_1	b_2	2
1	b_3	1	0
2	0	a_1	a_2

T20

$\to$	0	1	2
0	1	0	2
1	a_1	1	a_2
2	0	a_3	a_4

T21

$\to$	0	1	2
0	d_1	0	2
1	a_1	1	a_2
2	0	0	1

T22

→	0	1	2
0	0	0	2
1	a_1	1	a_2
2	0	b_1	1

T23

→	0	1	2
0	2	0	2
1	a_1	1	a_2
2	0	b_1	1

T24

→	0	1	2
0	0	b_1	2
1	a_1	1	0
2	0	a_2	1

T25

→	0	1	2
0	1	b_1	2
1	0	1	0
2	0	a_1	a_2

T26

→	0	1	2
0	2	b_1	2
1	0	1	0
2	0	a_1	1

T27

→	0	1	2
0	2	0	2
1	a_1	1	0
2	0	b_1	2

Proof Case (a) T16. The wff $(p \wedge \neg p) \to (q \vee \neg q)$ is M-valid in any implicative expansion of MK3, M, built upon any function in T16, for any propositional variables p, q. Cases (b) (T17), (c) (T18) and (d) (T19) are proved similarly: cases (b) and (c) (resp., (d)) using the wff $(p \vee \neg p) \to (q \vee \neg q)$ (resp., $(p \wedge \neg p) \to (q \wedge \neg q)$). Cases (e) (T20) through (l) (T27) are proved by using the wff $Tp \to Tq$, where the unary connective T is defined similarly as in Theorem 5.1: for any wff A, $TA = \bullet A \to \bullet A$ (T20, T25, T27) and $TA = \neg \bullet A \to \neg \bullet A$ (T21, T22, T23, T24 and T26).

Next, it is shown that the rest of the $\mathrm{T1}_1$-logics have the vsp.

Proposition 7.12 ($\mathbf{T1_1}$-logics with the vsp) *Let L be an implicative expansion of MK3 built upon any of the 126 functions in the tables T28 through T33. Then, L has the vsp.*

T28

→	0	1	2
0	d_1	0	2
1	a_1	1	a_2
2	0	0	d_2

T29

→	0	1	2
0	2	b_1	2
1	0	1	0
2	0	a_1	d_1

T30

→	0	1	2
0	0	0	2
1	a_1	1	a_2
2	0	b_1	0

T31

→	0	1	2
0	0	0	2
1	a_1	1	0
2	0	b_1	2

T32

→	0	1	2
0	2	0	2
1	a_1	1	a_2
2	0	b_1	0

T33

→	0	1	2
0	0	b_1	2
1	a_1	1	0
2	0	a_2	d_1

Proof Case (a) T28. Let M be an implicative expansion of MK3 built upon any function in T28. Suppose that there are wffs A, B such that $A \to B$ is M-valid but A and B do not share propositional variables. Let I be an M-interpretation assigning 0 (resp., 1) to each propositional variable in A (resp., B). Then, $I(A) \in \{0, 2\}$ and $I(B) = 1$, since $\{0, 2\}$ and $\{1\}$ are closed under $\to, \wedge, \vee$ and $\neg$. So, $I(A \to B) = 0$, contradicting the hypothesis.

Case (b) T29. Proof similar to case (a) by using now the fact $f_{\to}(1, 0) = f_{\to}(1, 2) = 0$.

Case (c) T30. Similarly as in case (a), we define an M-interpretation I such that $I(A) \in \{0, 2\}$ and $I(B) = 1$. Case (c1): If $I(A) = 0$, then $I(A \to B) = 0$, contradicting the hypothesis. Case (c2): $I(A) = 2$. Let I' be an M-interpretation exactly as I, except that for each propositional variable p in B,

$I'(p) = 0$. Clearly $I'(B) \in \{0, 2\}$ and $I'(A) = 2$, since A and B do not share propositional variables. Then, $I'(A \to B) = 0$, the hypothesis being again contradicted.

The rest of the cases are proved similarly by defining an M-interpretation I such that $I(A) \in \{0, 2\}$ and $I(B) = 1$, and a new interpretation I' such that $I'(B) \in \{0, 2\}$ (case (e) (T32)); (in cases (d) and (f), (T31), (T33), $I(A) = 1$, $I(B) \in \{0, 2\}$ and $I'(A) \in \{0, 2\}$).

8 Concluding remarks

The paper is ended with some concluding remarks. It would be interesting to investigate the following questions.

1. In [19], all unary and binary expansions of MK3 (2 is the only designated value) are characterized by means of natural deduction systems; in [9], a similar result is provided for the same type of expansions of MK3, this time with both 1 and 2 as designated values (cf. also [11]). Nevertheless, we lack a parallel characterization of 3-valued expansions of MK3 using Hilbert-style systems instead of natural deduction ones. In this regard, in [14] and [17] (cf. also [15]), Hilbert-type axiomatic systems are provided for all natural implicative expansions of MK3 in Tomova's sense. We wonder if this result can be extended to the new logics introduced in [16] and in the present paper and, more generally, to unary and binary expansions of MK3, similarly as it is the case with natural deduction systems in [9] and [19].
2. Priest's logic LP (cf. [12]) fails to have a natural conditional (both in Tomova's and in our definition of the notion), since the validity of Modus Ponens is not preserved in this logic. LP is one of the implicative expansions of MK3 contained in the general table $T1_1$ (cf. §6). We are interested in the question whether other elements in $T1_1$ are worthy of consideration when compared to LP. Also, we think that it is worth-while to further examine the properties of members in T1 (cf. Definition 2.6), in general.
3. Are there alternative structures to T1? That is, general tables with similar properties, i.e., functional inclusion of Ł3, with some members of these hypothetical structures being functionally complete or having the vsp and vsp-related properties or, finally, having other properties of interest?

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