A remark on functional completeness of binary expansions of Kleene's strong 3-valued logic

Gemma Robles* and José M. Méndez**
*Dpto. de Psicología, Sociología y Filosofía, Universidad de León
Campus de Vegazana, s/n, 24071, León, Spain
gemma.robles@unileon.es; http://grobv.unileon.es
ORCID: 0000-0001-6495-0388
**Universidad de Salamanca
Edificio FES, Campus Unamuno, 37007, Salamanca, Spain
sefus@usal.es; http://sites.google.com/site/sefusmendez
ORCID: 0000-0002-9560-3327

Abstract

A classical result by Słupecki states that a logic L is functionally complete for the 3-element set of truth-values THREE if, in addition to functionally including Łukasiewicz's 3-valued logic Ł3, what he names the 'T-function' is definable in L. By leaning upon this classical result, we prove a general theorem for defining binary expansions (i.e., expansions with a binary connective) of Kleene's strong logic that are functionally complete for THREE.

Keywords: 3-valued logic; Kleene's strong 3-valued logic; Łukasiewicz's 3-valued logic; functional completeness.

1 Introduction

Kleene's strong 3-valued matrix MK3 was introduced in [5] in the context of the treatment of partial recursive functions. The matrix MK3 (our label) can be defined as shown in Definition 2.3 below. The connectives in the propositional language are conjunction, disjunction and negation. We can take either 2 as the only designated value or else both 1 and 2. In the former case, 1 can be interpreted as neither truth nor falsity; in the latter, as both truth and falsity. The value 2 is, of course, truth, while 0 is falsity.

Two classical results on functional completeness by Słupecki, the first one in many-valued logic in general, the second one, in Łukasiewicz's 3-valued logic in particular, are the following:

Theorem 1.1 (On functional completeness in 3-valued logic) *Łukasiewicz's 3-valued logic Ł3 is not functionally complete for the 3-element set of truth-*

values THREE (cf. Definition 2.10 below). However, let L be a logic functionally equivalent to L3 (cf. Definition 2.9 below). If the unary connective T, given by the truth-table

	T
0	1
1	1
2	1

is definable in L, then L is complete for THREE (recall that in the present paper we use 2 *and* 1 *instead of* 1 *and* 1/2*, respectively; cf. Definition 2.3) (cf. [8]).*

Remark 1.2 (On the T-function) *Concerning the label T, Słupecki notes: "Following Prof. Dr. Łukasiewicz, this function will be referred to as the "tertium" and will be represented by Tp..." ([8], p. 10).*

Słupecki's results in [8] are expressed in Theorem 1.1 in our own terms. However, the main result in [9], here referred to as Theorem 1.3, is literally quoted ([9], p.39).

Theorem 1.3 (On functional completeness in many-valued logic) *If in a given many-valued logic all singular functions are defined, then this logic is a full system if and only if at least one of its primitive terms, being a binary functor, possesses an interpretative table in which at least one line and one column do not have all elements identical and in which all the values that in the considered many-valued logic may be taken by propositional variables are elements.*

The aim of the present paper is to prove a theorem similar to Theorem 1.3, this time for the case of binary expansions (i.e., expansions with a binary function) of MK3 and by leaning on the results recorded in Theorem 1.1.

In particular, we will prove facts (1) and (2) noted below. Let MŁ3 be the matrix determining Ł3 (cf. Theorem 3.5 below) and let M be a binary expansion of MK3.

1. If one of the negation functions $f_{\overset{\circ}{\neg}}$ and $f_{\overset{\bullet}{\neg}}$, given by the truth-tables

	$\overset{\circ}{\neg}$
0	2
1	0
2	0

	$\overset{\bullet}{\neg}$
0	2
1	2
2	0

   is definable in M, then M functionally includes MŁ3.

2. If one of $f_{\overset{\circ}{\neg}}$ and $f_{\overset{\bullet}{\neg}}$ is definable in M and f_* is a binary function in M such that one of the four following conditions (1)-(4) is fulfilled, then the connective T is definable in M and, consequently, M is complete for THREE: (1) $f_*(0,0) = 1$; (2) $f_*(0,2) = 1$; (3) $f_*(2,0) = 1$; (4) $f_*(2,2) = 1$.

In order to prove fact (1), we lean upon the following theorem proved in [7].

Theorem 1.4 (A class of exp. of MK3 funct. equiv. to MŁ3) *Consider the following general table T0 (a_i $(1 \leq i \leq 3) \in \{0, 1, 2\}$; $c_1 \in \{0, 1\}$; cf. Convention 3.1 below).*

T0

$\rightarrow$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	c_1	2

Let M be an expansion of MK3 built up by adding any of the 54 $f_{\rightarrow}$-functions contained in table T0. Then, M and MŁ3 are functionally equivalent.

In addition to prove facts (1) and (2), we shall apply them in defining a class of binary expansions of MK3 which are functionally complete for THREE. We have:

3. Let M be an expansion of MK3 built up adding the binary function f_*. If one of the ensuing conditions (a) or (b) is fulfilled, then M functionally includes MŁ3. (a) $f_*(1, 1) = 2$ and either $f_*(2, 0) = 0$ or else $f_*(2, 2) = 0$; (b) $f_*(1, 1) = 0$ and either $f_*(2, 0) = 2$ or $f_*(2, 2) = 2$.

4. Let C be the class of expansions of MK3 defined in (3). We select the subset C' of C formed by the elements that are functionally complete for THREE. As pointed out above, we prove functional completeness for THREE by showing how to define the unary connective T in each one of the elements of C'.

We note that the possible interest of the classes C and C' in 4 is not merely formal: many of the binary functions in them can be considered implication functions.

The paper is organized as follows. In §2, we record some preliminary definitions as used in the paper. Also, three lemmas to be used in the sequel are proved. In §3, facts (1) and (2) commented upon above are established. In §4, we prove facts (3) and (4): we define a class of functionally complete for THREE expansions of MK3 by leaning upon facts (1) and (2) proven in §3. Finally, in §5, the paper is concluded with some remarks on the results obtained.

2 Preliminary notions and lemmas

In this section, we record some preliminary notions as used in the present paper (of course, there are alternative definitions of these notions).

Definition 2.1 (Language) *The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.*

Definition 2.2 (Logical matrix) *A (logical) matrix is a structure* $(\mathcal{V}, D, \mathtt{F})$ *where (1)* $\mathcal{V}$ *is a set of (truth) values; (2)* D *is a non-empty proper subset of* $\mathcal{V}$ *(the set of designated values); and (3)* $\mathtt{F}$ *is the set of* n*-ary functions on* $\mathcal{V}$ *such that for each* n*-ary connective* c *(of the propositional language in question), there is a function* $f_c \in \mathtt{F}$ *such that* $f_c : \mathcal{V}^n \to \mathcal{V}$*. An M-interpretation is a function from the set of all wffs to* $\mathcal{V}$ *according to the functions in* $\mathtt{F}$*.*

Next, Kleene's strong 3-valued matrix is defined. Notice that we can choose only 2 or else 1 and 2 as designated values. (Kleene uses $\mathfrak{t}$, $\mathfrak{f}$ and $\mathfrak{u}$ instead of 2, 0 and 1, respectively. The latter have been chosen in order to use the tester in [3], in case the reader needs one. Also, to put in connection the results in the present paper with previous work by us.)

Definition 2.3 (Kleene's strong 3-valued matrix) *The propositional language consists of the connectives* $\wedge, \vee, \neg$*. Kleene's strong 3-valued matrix, MK3 (our label), is the structure* $(\mathcal{V}, D, \mathtt{F})$ *where (1)* $\mathcal{V} = \{0, 1, 2\}$*; (2)* $D = \{1, 2\}$ *or* $D = \{2\}$*; (3)* $\mathtt{F} = \{f_\wedge, f_\vee, f_\neg\}$ *where* $f_\wedge$ *and* $f_\vee$ *are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and* $f_\neg$ *is an involution with* $f_\neg(2) = 0, f_\neg(0) = 2$ *and* $f_\neg(1) = 1$*. We display the tables for* $\wedge$*,* $\vee$ *and* $\neg$*:*

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\neg$	0
0	2
1	1
2	0

The notion of an MK3-interpretation is defined according to the general Definition 2.2.

The logic determined by MK3 can be named here $K3^1$ (only one designated value) or $K3^2$ (two designated values) (cf. [2], §4 on these logics). Then, all logics defined in this paper are expansions of either $K3^1$ or else $K3^2$.

Remark 2.4 (On designated values in MK3) *Kleene does not seem to have considered designated values in [5], §64, although he remarks: "The third "truth value"* $\mathfrak{u}$ *is thus not on a par with the other two* $\mathfrak{t}$ *and* $\mathfrak{f}$ *in our theory. Consideration of its status will show that we are limited to a special kind of truth value." ([5], p. 333). Priest logic LP (cf. [6]) is essentially the result of taking* $\mathfrak{t}$ *and* $\mathfrak{u}$ *as designated values in Kleene's 3-valued logic. According to Karpenko (cf. [4], p. 83), the idea of defining such a logic first appeared in [1].*

Below, some additional preliminary definitions are stated.

Definition 2.5 (Functions definable in a matrix) *Let* M *be a matrix* $(\mathcal{V}, D, \mathtt{F})$*. An* n*-ary function* f_* *is definable in* M *if for all* $a_1, ...a_n \in \mathcal{V}$ *there is some* fp *and* $e_{1,...}, e_m$ *such that* $fp \in \mathtt{F}$ *and, for all* i $(1 \leq i \leq m)$*, either* $e_i \in \mathcal{V}$ *or else* $e_i \in \mathtt{F}$ *and* $f * (a_1, .., a_n) = fp(e_i, ..., e_m)$*.*

Definition 2.6 (EMK3s considered in the paper) *Let f_* be a unary or a binary function defined in $\mathcal{V} = \{0, 1, 2\}$. Addition of f_* to the set* $\mathtt{F}$ *in MK3 is an expansion of MK3 if f_* is not definable in MK3 (the abbreviation EMK3 is often used for referring to an expansion of MK3).*

Definition 2.7 (Implicative expansions of MK3) *Let M be an EMK3 defined by adding the function f_*. M is an implicative expansion of MK3 if f_* can be considered an implication function in some sense of the term 'implication'.*

Definition 2.8 (Functional inclusion) *Let M and M′ be expansions of MK3. M is functionally included in M′ if f is definable in M′ for all $f \in \mathtt{F}$ in M.*

Definition 2.9 (Functional equivalence) *Let M and M′ be expansions of MK3. M and M′ are functionally equivalent if M and M′ are functionally included in each other.*

Definition 2.10 (Functional completeness) *Let $\mathcal{V}$ be a set of n elements (truth-values). A matrix M is functionally complete for $\mathcal{V}$ if every m-ary ($1 \leq m \leq n$) function definable in $\mathcal{V}$ is definable in M.*

Next, we prove some preliminary lemmas. We note the following remark on the proofs to follow in this and the rest of the sections of the paper.

Remark 2.11 (On displaying proofs of definability) *Let f_* be a function defined in $\mathcal{V} = \{0, 1, 2\}$. In what follows, f_* is usually represented by means of a truth-table t_* (or simply $*$), as for instance, it is the case with $\wedge, \vee$ and $\neg$ in MK3 (cf. Definition 2.3). In addition, by k_* (or simply $*$), we refer to the connective defined by f_*, represented, as said, in the table t_*. Now, let M be MK3 or an expansion of it. The proof that a given unary or binary function f_* is definable in M is easily visualized by using k_* and the connectives corresponding to the functions in M needed in the proof in question. Consequently, this way of displaying proofs of definability of functions is followed in the sequel.*

Lemma 2.12 (Two interdefinable negation connectives) *Consider the negation functions $f_{\overset{\bullet}{\neg}}$ and $f_{\overset{\circ}{\neg}}$, given by the truth-tables*

	$\overset{\bullet}{\neg}$
0	2
1	2
2	0

	$\overset{\circ}{\neg}$
0	2
1	0
2	0

Given MK3, $f_{\overset{\bullet}{\neg}}$ and $f_{\overset{\circ}{\neg}}$ are definable from each other.

Proof. For any wff A, we set $\overset{\bullet}{\neg}A = \neg\overset{\circ}{\neg}\neg A$ and $\overset{\circ}{\neg}A = \neg\overset{\bullet}{\neg}\neg A$. ∎

Lemma 2.13 (An implication function) *The implication function $f_{\overset{\bullet}{\rightarrow}}$, given by the truth-table*

$\overset{\bullet}{\rightarrow}$	0	1	2
0	2	2	2
1	1	1	2
2	0	1	2

is definable in any EMK3.

Proof. For any wffs A, B, we set $A \overset{\bullet}{\rightarrow} B = \neg A \vee B$, for any wffs A, B. Actually, $\overset{\bullet}{\rightarrow}$ is the implication of Kleene's 3-valued logic, which is defined exactly as in the proof just given (cf. [5], p.336). ■

Lemma 2.14 (Some add. func. def. in any EMK3 with $f_{\overset{\circ}{\neg}}$) *Let M be any EMK3 where the negation function $f_{\overset{\circ}{\neg}}$, given by the truth table is definable (cf. Lemma 2.12). Then, the functions f_1, f_2, f_3 and f_4, given by the truth tables*

t1	0	1	2
0	2	2	2
1	2	0	0
2	2	0	0

t2	0	1	2
0	2	2	2
1	2	1	2
2	2	2	2

t3	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

t4	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

are also definable in M.

Proof. (a) For any wffs A, B, we set $A\ k_1\ B = \overset{\circ}{\neg} A \vee \overset{\circ}{\neg} B$. (b) $A\ k_2\ B = \neg(A \wedge B) \vee (A \vee B)$. (c) $A\ k_3\ B = (A \wedge B) \wedge \overset{\circ}{\neg} A$. (d) $A\ k_4\ B = \neg(A\ k_2\ B)$. ■

In the next section, we prove facts (1) and (2) recorded in the introduction to the paper.

3 The main theorems

We begin by stating some useful definitional conventions frequently employed throughout the paper.

Convention 3.1 (Sets built up from $\mathcal{V} = \{0, 1, 2\}$) *In the general tables displayed in the paper, sets of truth-values in $\mathcal{V} = \{0, 1, 2\}$ are represented similarly as in T0 above. That is, a_i $(1 \leq i \leq 9) \in \{0, 1, 2\}$; b_i $(1 \leq i \leq 9) \in \{1, 2\}$; c_i $(1 \leq i \leq 9) \in \{0, 1\}$ and d_i $(1 \leq i \leq 9) \in \{0, 2\}$.*

Convention 3.2 (Proof of functional equivalence) *Let M and M′ be two expansions of MK3. We show that M is functionally included in M′ by proving that the tables representing the new functions added to MK3 in order to build up M are definable in M′. Consequently, the proofs of definability of some table by another given in the following pages have to be understood in this sense and as given in the context of two or more expansions of MK3 (cf. Remark 2.11).*

Convention 3.3 (Prop. of a gen. table; definable tables & connectives) *Let Tn be a general table as, say, T0 or T1 (cf. Definition 3.7 below). For instance, we say that T1 is included in (or definable from) T0 meaning that each expansion of MK3 built by adding any function in T1 is functionally included in each expansion of MK3 built by adding any function in T0. The same manner of speaking can be used in the case of a general table and a particular one, say, the characteristic implication table tŁ3 of Ł3. Also, we can say, for instance, that a given connective c is "definable from" T1, meaning that c is definable in each expansion of MK3 built up by adding any of the functions in T1. Finally, sometimes we say that P is a property of, for instance, T0, meaning that it is a property of each expansion of MK3 definable from T0, as the latter notion has been defined above.*

Convention 3.4 (Connectives and tables) *Let tn be a truth-table. By k_n, we refer to the connective defined by it, as pointed out in Remark 2.11. Then, given the binary connectives k_i and k_j and additional connective $\overset{m}{\to}$ (defined by table tm), by ti ∪ tj (resp., ti ∩ tj, ti $\overset{m}{\supset}$ tj, ¬ti) we refer, for any wffs A, B, to the table given by the wff $(A\ k_i\ B) \vee (A\ k_j\ B)$ (resp., $(A\ k_i\ B) \wedge (A\ k_j\ B)$, $(A\ k_i\ B) \overset{m}{\to} (A\ k_j\ B)$, $\neg(A\ k_i\ B)$). A similar convention is used in the case of general tables in their relation to either other general tables or else to a particular table defining some connective.*

Now, we have:

Theorem 3.5 (Impl. EMK3s func. including MŁ3 I) *Let M be any EMK3 in which the negation function $f_{\overset{\bullet}{\neg}}$ is definable (cf. Lemma 2.12). Then, M functionally includes the matrix MŁ3 determining the logic Ł3.*

Proof. Consider the following implication function $f_{\overset{\circ}{\to}}$, given by the truth-table

$\overset{\circ}{\to}$	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

The function $f_{\overset{\circ}{\to}}$ is definable in M by setting, for any wffs A, B, $A \overset{\circ}{\to} B = (A \overset{\bullet}{\to} B) \vee \overset{\bullet}{\neg} A$ (cf. Lemmas 2.12 and 2.13). But, as recalled above (cf. Theorem 1.4), the implicative EMK3 built up by adding $f_{\overset{\circ}{\to}}$ to MK3 is one of the 54 implicative expansions of MK3 definable from the general table T0

T0

$\to$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	c_1	2

shown functionally equivalent to MŁ3, i.e., the EMK3 defined by adding to MK3 the characteristic implication table of Ł3,

tŁ3	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

in [7]. (By the way, notice that $f_{\overset{\circ}{\neg}}$ is definable as follows: $\overset{\circ}{\neg}A = \neg(\neg A \overset{\circ}{\to} A)$ for any wff A; also notice that a direct definition of tŁ3 can be given as follows: $(A \overset{Ł3}{\to} B) = (A \overset{\bullet}{\to} B) \vee \overset{\bullet}{\neg}(A\ k_2\ B)$ —cf. Lemma 2.14. ■

Given Theorem 3.5 and Lemma 2.12, we have the following corollary.

Corollary 3.6 (Impl. EMK3s func. including MŁ3 II) *Let M be any EMK3 in which the negation function $f_{\overset{\circ}{\neg}}$ is definable (cf. Lemma 2.12). Then, M functionally includes MŁ3.*

Proof. Immediate by Theorem 3.5 and Lemma 2.12: $f_{\overset{\bullet}{\neg}}$ is definable in M. ■

In what follows, we proceed to show how to define Słupecki's unary function T in binary expansions of MK3. We use Conventions 3.1 through 3.4 in this and the rest of the sections of the paper. Firstly, we set the following definition.

Definition 3.7 (The general tables T1-T4 and T1$_1$-T4$_1$) *The general tables T1, T2, T3 and T4, and T1$_1$, T2$_1$, T3$_1$ and T4$_1$ are the following.*

T1

$*$	0	1	2
0	1	a_1	a_2
1	a_3	a_4	a_5
2	a_6	a_7	a_8

T2

$*$	0	1	2
0	a_1	a_2	1
1	a_3	a_4	a_5
2	a_6	a_7	a_8

T3

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	a_5	a_6
2	1	a_7	a_8

T4

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	a_5	a_6
2	a_7	a_8	1

T1$_1$

$*$	0	1	2
0	1	a_1	a_2
1	a_3	1	a_4
2	a_5	a_6	a_7

T2$_1$

$*$	0	1	2
0	a_1	a_2	1
1	a_3	1	a_4
2	a_5	a_6	a_7

T3$_1$

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	1	a_5
2	1	a_6	a_7

T4$_1$

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	1	a_5
2	a_6	a_7	1

It is clear that T1$_1$, T2$_1$, T3$_1$ and T4$_1$ are included in T1, T2, T3 and T4, respectively: they are the result of selecting the value 1 among the three possible ones for $f_*(1,1)$.

Concerning T1$_1$, T2$_1$, T3$_1$ and T4$_1$, we prove the ensuing proposition.

Proposition 3.8 (Tables $T1_1$-$T4_1$ and the T-function) *Słupecki's T-function is definable in $T1_1$, $T2_1$, $T3_1$ and $T4_1$. That is, let M be an EMK3 built up by adding any of the binary functions in $T1_1$, $T2_1$, $T3_1$ or $T4_1$. Then, the T-function is definable in M.*

Proof. We use Remark 2.11. In what follows, A represent any wff; also, $*_i$ represents the connective in the table Ti_1 ($1 \leq i \leq 4$). (a) $T1_1$-$T4_1$ are definable from each other: $A *_1 B = A *_2 \neg B$; $A *_1 B = \neg A *_3 B$; $A *_1 B = \neg A *_4 \neg B$; $A *_2 B = A *_1 \neg B$; $A *_3 B = \neg A *_1 B$; $A *_4 B = \neg A *_1 \neg B$. (b) It is shown that T is definable in $T1_1$. $a_7 = 1$: $TA = A *_1 A$; $a_7 = 0$: $TA = (A *_1 A) \vee (\neg A *_1 \neg A)$; $a_7 = 2$: $TA = (A *_1 A) \wedge (\neg A *_1 \neg A)$. ■

Next, we prove a general theorem on functional completeness.

Theorem 3.9 (Tables functionally complete for THREE) *Consider the general tables T1, T2, T3 and T4 in Definition 3.7. And let $T1_a$, $T2_a$, $T3_a$ and $T4_a$ be tables included in (or equivalent to) T1, T2, T3 and T4, respectively. If one of the negation functions $f_{\underset{\neg}{\circ}}$ or $f_{\underset{\neg}{\bullet}}$ is definable in Ti_a ($1 \leq i \leq 4$), then Ti_a is functionally complete for THREE. That is, if M is an EMK3 built up by adding to MK3 any of the binary functions in Ti_a, then M is functionally complete for THREE, provided $f_{\underset{\neg}{\circ}}$ or $f_{\underset{\neg}{\bullet}}$ is definable in M (cf. Lemma 2.12 on the negation functions $f_{\underset{\neg}{\circ}}$ and $f_{\underset{\neg}{\bullet}}$).*

Proof. The proof leans on Słupecki's classical result recorded in Theorem 1.1: Let M be an EMK3 functionally including MŁ3 where the T-function is definable. Then, M is functionally complete for THREE.

Suppose then that either $f_{\underset{\neg}{\circ}}$ or $f_{\underset{\neg}{\bullet}}$ is definable in Ti_a ($1 \leq i \leq 4$). Then, we have (1) Ti_a functionally includes MŁ3 (cf. Theorem 3.5 and Corollary 3.6); and (2) tables t2 and t4

t2	0	1	2
0	2	2	2
1	2	1	2
2	2	2	2

t4	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

are definable from Ti_a (cf. Lemma 2.12 and 2.14). Moreover, (3). Let $T1_{a1}$, $T2_{a1}$, $T3_{a1}$ and $T4_{a1}$ be the result of changing $f_*(1,1) = 2$ for $f_*(1,1) = 1$ in $T1_a$, $T2_a$, $T3_a$ and $T4_a$, respectively. Then, $T1_{a1}$, $T2_{a1}$, $T3_{a1}$ and $T4_{a1}$ are definable from $T1_a$, $T2_a$, $T3_a$ and $T4_a$, respectively, as we proceed to prove. We examine the three possible truth values the pair $\langle 1,1 \rangle$ can take for f_* in Ti_a in order to show that in all cases Ti_{a1} follows from Ti_a ($1 \leq i \leq 4$). (a) $f_*(1,1) = 1$. Then, $T1_a$, $T2_a$, $T3_a$ and $T4_a$ are in fact $T1_{a1}$, $T2_{a1}$, $T3_{a1}$ and $T4_{a1}$, respectively. (b) $f_*(1,1) = 0$. Then, T$i_a \cup$ t4 = Ti_{a1}. (c) $f_*(1,1) = 2$. Then, T$i_a \cap$ t2 = Ti_{a1}.

Consequently, it follows from facts (1), (2) and (3) proven above that Ti_a ($1 \leq i \leq 4$) is functionally complete for THREE, provided $f_{\underset{\neg}{\circ}}$ or $f_{\underset{\neg}{\bullet}}$ is definable in Ti_a. ■

Tables T1-T4 enclose a set of possible functionally complete for THREE EMK3s, a subset of which is defined in the pages to follow. Meanwhile, we end the section with the ensuing remark where a set of EMK3s not functionally complete for THREE is recorded.

Remark 3.10 (A set of EMK3s not funct. compl. for THREE) *Consider the general table* $T0'$

$T0'$.

	0	1	2
0	d_1	a_1	d_2
1	a_2	a_3	a_4
2	d_3	a_5	d_4

Let M be an EMK3 built up by adding to MK3 any of the 3888 functions in this table. Then, M is not functionally complete for THREE: it is not possible to define a function taking the value 1 *if it only has* 0 *and* 2 *as its arguments (cf. the classical paper [8] by Słupecki). (We note that some of the functions in* $T0'$ *are definable from MK3. For instance, consider the following wffs:* $\neg A \vee B$, $A \vee \neg B$, $\neg A \vee \neg B$, $\neg A \wedge B$, $A \wedge \neg B$, $\neg A \wedge \neg B$. *The functions defined by each one of these wffs is included in* $T0'$*.)*

4 A class of functionally complete for THREE expansions of MK3

By using Theorem 3.9, in the sequel, we define a class of EMK3s functionally complete for THREE. In the first place, we remark a class of EMK3s where the function $f_{\stackrel{\bullet}{\neg}}$ is definable.

Proposition 4.1 (A class of EMK3s where $f_{\stackrel{\bullet}{\neg}}$ is definable) *Consider the 8748 binary functions included in the general tables T5, T6, T7 and T8*

T5

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	2	a_5
2	0	a_6	a_7

T6

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	2	a_5
2	a_6	a_7	0

T7

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	0	a_5
2	2	a_6	a_7

T8

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	0	a_5
2	a_6	a_7	2

The negation function $f_{\stackrel{\bullet}{\neg}}$ *is definable in Ti* $(5 \leq i \leq 8)$.

Proof. For any wff A, we set (a) T5: $\stackrel{\bullet}{\neg}A = (A\ k_* \neg A) \vee \neg A$; (b) T6: $\stackrel{\bullet}{\neg}A = (A\ k_*\ A) \vee \neg A$; (c) T7: $\stackrel{\bullet}{\neg}A = (A\ k_* \neg A) \stackrel{\bullet}{\to} \neg A$. (d) T8: $\stackrel{\bullet}{\neg}A = (A\ k_*\ A) \stackrel{\bullet}{\to} \neg A$ ($\stackrel{\bullet}{\to}$ is the conditional defined in Lemma 2.13). ■

We note the following corollary.

Corollary 4.2 (T5, T6, T7 and T8 funct. include MŁ3) *Each one of the tables T5, T6, T7 and T8 functionally includes the matrix MŁ3 determining the logic Ł3.*

Proof. Immediate by Proposition 4.1 and Theorem 3.5. ■

The 8748 binary functions displayed in Proposition 4.1 are not different from each other. The 7290 functions different from each other in tables T5, T6, T7 and T8 are collected in Proposition 4.3.

Proposition 4.3 (Functions diff. from each other in Prop. 4.1) *The general tables T5, T6′, T7 and T8′ collect all functions different from each other in tables T5, T6, T7 and T8 in Proposition 4.1. Tables T6′ and T8′ are*

T6′

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	2	a_5
2	b_1	a_6	0

T8′

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	0	a_5
2	c_1	a_6	2

Proof. It is immediate by inspection. Notice that the general tables T6″ and T8″

T6″

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	2	a_5
2	0	a_6	0

T8″

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	0	a_5
2	2	a_6	2

(the former included in T6′ and the latter in T8′) are included in T5 and T7, respectively. ■

Next, the class of all functionally complete for THREE EMK3s contained in tables T5, T6′, T7 and T8′ in Proposition 4.3 is defined.

Proposition 4.4 (A class of functionally complete for THREE EMK3s) *Consider the following general tables* $T5_a$, $T5_b$, $T5_c$ *(included in T5),* $T6'_a$, $T6'_b$, $T6'_c$ *(included in T6′),* $T7_a$, $T7_b$, $T7_c$ *(included in T7), and* $T8'_a$, $T8'_b$ *and* $T8'_c$ *(included in T8′).*

$T5_a$

$*$	0	1	2
0	1	a_1	a_2
1	a_3	2	a_4
2	0	a_5	a_6

$T5_b$

$*$	0	1	2
0	d_1	a_1	1
1	a_2	2	a_3
2	0	a_4	a_5

$T5_c$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	2	a_3
2	0	a_4	1

$T6'_a$

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	2	a_5
2	1	a_6	0

$T6'_b$

$*$	0	1	2
0	1	a_1	a_2
1	a_3	2	a_4
2	2	a_5	0

$T6'_c$

$*$	0	1	2
0	d_1	a_1	1
1	a_2	2	a_3
2	2	a_4	0

$T7_a$

$*$	0	1	2
0	1	a_1	a_2
1	a_3	0	a_4
2	2	a_5	a_6

$T7_b$

$*$	0	1	2
0	d_1	a_1	1
1	a_2	0	a_3
2	2	a_4	a_5

$T7_c$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	0	a_3
2	2	a_4	1

$T8'_a$

$*$	0	1	2
0	a_1	a_2	a_3
1	a_4	0	a_5
2	1	a_6	2

$T8'_b$

$*$	0	1	2
0	1	a_1	a_2
1	a_3	0	a_4
2	0	a_5	2

$T8'_c$

$*$	0	1	2
0	d_1	a_1	1
1	a_2	0	a_3
2	0	a_4	2

Any functionally complete for THREE expansion of MK3 definable from tables T5, T6′, T7 and T8′ is contained in one of the 12 tables displayed above.

Proof. Let us generally refer by T^{12} to the 12 general tables displayed above. And let Tm be a table in T^{12}.

(I) T^{12} is functionally complete for THREE:

We have (a) Tm is included in at least one of the tables T1, T2, T3 and T4 in Definition 3.7. (b) Tm is included in at least one of the tables T5, T6, T7 and T8 in Proposition 4.1. Consequently, it follows from Theorem 3.9 that Tm is functionally complete for THREE.

(II) T^{12} collects the 5346 functionally complete for THREE EMK3s definable from T5, T6, T7 and T8 in Proposition 4.1:

In order to prove (II), it suffices to note that the ensuing general tables $\mathrm{T5}_d$, $\mathrm{T6}'_d$, $\mathrm{T7}_d$ and $\mathrm{T8}'_d$

$\mathrm{T5}_d$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	2	a_3
2	0	a_4	d_3

$\mathrm{T6}'_d$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	2	a_3
2	2	a_4	0

$\mathrm{T7}_d$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	0	a_3
2	2	a_4	d_3

$\mathrm{T8}'_d$

$*$	0	1	2
0	d_1	a_1	d_2
1	a_2	2	a_3
2	0	a_4	0

(included in T5, T6′, T7 and T8′, respectively) are in addition included in the general table recorded in Remark 3.10, shown to be not functionally complete for THREE. ■

5 Concluding remarks

The paper is ended with some concluding remarks. We have shown that MŁ3 is included in any class of EMK3s where the negation function $f_{\underset{\bullet}{\neg}}$ (or the equivalent one $f_{\underset{\circ}{\neg}}$) is definable (Theorem 3.5, Corollary 3.6). Then, we have defined a class of EMK3s where Słupecki's T-function is definable (Proposition 3.9). It follows from Słupecki's classical result in [8] that the intersection of both classes is composed of functionally complete for THREE EMK3s. But the class of EMK3s in Proposition 4.1 functionally including MŁ3 as well as the ones functionally complete for THREE in Proposition 4.4 are remarked in the paper as a way of an example. By using Theorem 3.5 (or Corollary 3.6) and Theorem 3.9, it is not difficult to define alternative classes functionally including MŁ3 and/or functionally complete for THREE. Let us propose an example. Consider the 324 tables contained in the following general tables T9, T10, T11 and T12.

T9

$*$	0	1	2
0	a_1	a_2	2
1	a_3	2	0
2	2	a_4	1

T10

$*$	0	1	2
0	a_1	a_2	0
1	a_3	0	2
2	0	a_4	1

T11

$*$	0	1	2
0	0	a_1	a_2
1	2	0	a_3
2	1	a_4	0

T12

$*$	0	1	2
0	2	a_1	a_2
1	0	2	a_3
2	1	a_4	2

These four tables are not included in T^{12} (Proposition 4.4) and are functionally complete for THREE. We have (a) T9 and T10 (resp., T11 and T12) are definable from each other: T9 = ¬T10, T10 = ¬T9, T11 = ¬T12 and T12 = ¬T11; (b) $f_{\overset{\circ}{\neg}}$ is definable in T9 and T11 as follows: for any wff A, we set $\overset{\circ}{\neg}A = [A *_9 (A *_9 \neg A)] \wedge \neg A$; $\overset{\circ}{\neg}A = \neg[A *_{11} (A *_{11} A)] \wedge \neg A$; (c) The T-operator is definable in T9 and T11 as follows. Firstly, remark that $\mathrm{T9}_1$ and $\mathrm{T11}_1$ are definable from T9 and T11, respectively (cf. Theorem 3.9). Next, let $*_9$ be the connective in table $\mathrm{T9}_1$; $*_{11}$, the connective in table $\mathrm{T11}_1$, then for any wff A, $TA = \neg(A *_9 A) *_9 \neg(A *_9 A)$ if $a_1 = 0$, and $TA = (A *_9 A) *_9 (A *_9 A)$ if $a_1 = 2$. On the other hand, $TA = \neg(A *_{11} A) *_{11} (A *_{11} A)$; (d) finally, Słupecki's result in [8] together with Corollary 3.6 is applied.

Funding. - This work was supported by the Spanish Ministry of Economy, Industry and Competitiveness [FFI2017-82878-P]. - We sincerely thank an anonymous referee of the IGPL for her (his) careful reading of the manuscript and comments and suggestions resulting in an improvement of the first draft of the paper.

References

[1] Asenjo, F. G. (1953). La idea de un cálculo de antinomias. *Seminario Matemático*, Universidad Nacional de La Plata.

[2] Dunn, J. M. (2000). Partiality and its Dual. *Studia Logica*, 65, 5-40, https://doi.org/10.1023/A:1026740726955.

[3] González, C. (2012). *MaTest.* Retrieved from http://ceguel.es/matest (last accessed 31/01/2020).

[4] Karpenko, A. S. (1999). Jaśkowski's criterion and three-valued paraconsistent logics. *Logic and Logical Philosophy*, 7, 81-86. https://doi.org/10.12775/LLP.1999.006

[5] Kleene, S. C. (1952). *Introduction to Metamathematics*. North Holland. Reprinted Ishi Press, 2009.

[6] Priest, G. (1979). Logic of Paradox. *Journal of Philosophical Logic*, 8(1), 219-241.

[7] Robles, G., Méndez, J. M. (2019). The class of all natural implicative expansions of Kleene's strong logic functionally equivalent to Łukasiewicz's 3-valued logic Ł3. *Journal of Logic, Language and Information*, Online First, doi: 10.1007/s10849-019-09306-2.

[8] Słupecki, J. (1936). Der volle dreiwertige Aussagenkalkül. *Comptes Rendus Des Séances de La Société Des Sciences et Des Lettres de Varsovie*, Classe III, vol. 29, 9–11.

[9] Słupecki, J. (1972). A criterion of fullness of many-valued systems of propositional logic. *Studia Logica*, 30(1), 153–157. https://doi.org/10.1007/BF02120845.