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A basic dual intuitionistic logic and some of its extensions included in $\mathbf{G3_{DH}}$

Gemma Robles · José M. Méndez

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Abstract The logic DHb is the result of extending Sylvan and Plumwood's minimal De Morgan logic $\mathrm{B_M}$ with a dual intuitionistic negation of the type Sylvan defined for the extension $\mathrm{CC}\omega$ of da Costa's paraconsistent logic $\mathrm{C}\omega$. We provide Routley-Meyer ternary relational semantics with a set of designated points for DHb and a wealth of its extensions included in $\mathrm{G3_{DH}}$, the expansion of $\mathrm{G3_+}$ with a dual intuitionistic negation of the kind considered by Sylvan ($\mathrm{G3_+}$ is the positive fragment of Gödelian 3-valued logic G3). All logics in the paper are paraconsistent.

Keywords Dual intuitionistic logics · De Morgan logics · paraconsistent logics · Routley-Meyer ternary relational semantics

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1 Introduction

The aim of this paper is to introduce the logic DHb and a wealth of its extensions. The kind of negation these logics enjoy will be generally referred to

Gemma Robles
Dpto. de Psicología, Sociología y Filosofía, Universidad de León
Campus de Vegazana, s/n
24071, León, Spain
http://grobv.unileon.es
E-mail: gemma.robles@unileon.es
ORCID: 0000-0001-6495-0388

José M. Méndez
Universidad de Salamanca
Campus Unamuno, Edificio FES
37007, Salamanca, Spain
http://sites.google.com/site/sefusmendez
E-mail: sefus@usal.es
ORCID: 0000-0002-9560-3327

as DH-negation, DH-logics being the general term used to mention DHb and its extensions defined in the sequel. DH-negation can be considered as a dual intuitionistic negation in some sense to be explained in what follows (H stands for Heyting; DH for dual H-negation).

DHb is an extension of Sylvan and Plumwood's minimal De Morgan logic B_M, in its turn, an expansion of Routley and Meyer's basic positive logic B_+ (cf. Definitions 2.4 and 2.6). DH-logics are defined by using the list of theses and rules in Lemma 2.13. Of these, a32-a44 concern negation, while a1-a31 are formulated in the negationless logical language.

All DH-logics are endowed with an unreduced Routley-Meyer ternary relational semantics (RM-semantics) with a set of designated points (cf. [14] and [2]).

Concerning da Costa's paraconsistent logic Cω (cf. [3]), system to which we will return below, Richard Sylvan (*né* Routley) notes that "Cω is in certain respects the dual of intuitionistic logic" (cf. [16], p. 48). In particular, if a semantical point of view is adopted, Sylvan notes ([16], p. 49) "whereas intuitionism is essentially focused on evidentially incomplete situations excluding inconsistent situations, the C-systems admit inconsistent situations but remove incomplete situations". Well then, contrary to what happens in Kripke models for intuitionistic logic, the models in the RM-semantics we define for the DH-logics are composed exclusively of complete though not necessarily consistent elements, unlike it is the case in standard RM-semantics, where the elements can be incomplete, inconsistent or both.

But this duality can also take a proof-theoretical shape. Intuitionistic logic rejects the "Conditioned Principle of Excluded Middle" (PEM), $C \to (A \vee \neg A)$, and "Double Negation Elimination" (DNE), $\neg\neg A \to A$, but accepts "E Contradictione Quodlibet" (ECQ), $(A \wedge \neg A) \to B$, and "Double Negation Introduction" (DNI), $A \to \neg\neg A$, while Cω does the reverse (cf. [16], pp. 48-49). Well then, DH-logics work exactly as the C-systems in this respect: they assert CPEM and DNE, but reject ECQ and DNI.

Thus, we see, both from a semantical and a proof-theoretical standpoint, DH-logics are the dual of intuitionistic logics in a similar sense to which da Costa's C-systems, in general, and Cω, in particular, are.

But let us elaborate on the question. Consider the ensuing theses and rules of classical propositional logic (cf. Definition 2.1 on the logical language used in the paper; also the theses and rules in Lemma 2.13):

b1. $(A \to B) \to (\neg B \to \neg A)$
b2. $(A \to \neg B) \to (B \to \neg A)$
b3. $(\neg A \to B) \to (\neg B \to A)$
b4. $(\neg A \to \neg B) \to (B \to A)$
b1$'$. $A \to B \Rightarrow \neg B \to \neg A$
b2$'$. $A \to \neg B \Rightarrow B \to \neg A$
b3$'$. $\neg A \to B \Rightarrow \neg B \to A$
b4$'$. $\neg A \to \neg B \Rightarrow B \to A$

b5. $A \to \neg\neg A$
b6. $\neg\neg A \to A$
b7. $(A \wedge \neg A) \to B$
b8. $B \to (A \vee \neg A)$
b9. $(\neg A \vee \neg B) \to \neg(A \wedge B)$
b10. $(\neg A \wedge \neg B) \to \neg(A \vee B)$
b11. $\neg(A \wedge B) \to (\neg A \vee \neg B)$
b12. $\neg(A \vee B) \to (\neg A \wedge \neg B)$
b13. $(A \vee B) \to \neg(\neg A \wedge \neg B)$
b14. $(A \wedge B) \to \neg(\neg A \vee \neg B)$
b15. $\neg(\neg A \wedge \neg B) \to (A \vee B)$
b16. $\neg(\neg A \vee \neg B) \to (A \wedge B)$

b1-b4 are the contraposition axioms, and b1′-b4′ the corresponding contraposition rules. b5 (resp., b6) is the DNI (resp., DNE) axiom; and b7 (resp., b8) is the ECQ (resp., CPEM) axiom. b9-b12 are the De Morgan laws, and, finally, b13-b16 are the laws of interdefinition between conjunction and disjunction. Theses b1, b2 (so, rules b1′, b2′), b5, b7, b9, b10, b12, b13 and b14 are provable in intuitionistic logic, the rest of theses and rules b3′ and b4′ are not. Then, DHb and DH-logics in general lack b1, b3, b2′, b4′ (so, b2, b4), b5, b7, b13 and b14 (cf. Remark 2.15); but it has to be remarked that in addition to b3′, b6, b8, b11, b15 and b16, they also have b1′ and b9-b12 (that is, all De Morgan laws).

Among the DH-logics there are some extensions of the aforementioned da Costa's paraconsistent logic Cω that have to be remarked. Let us briefly discuss the question.

In [16], Sylvan notes that Cω lacks the "replacement of equivalents" theorem (RE), since it lacks the contraposition rule (Con) (i.e., b1 above: $A \to B \Rightarrow \neg B \to \neg A$). He proposes then to extend Cω with Con. The result is labelled CCω. Sylvan remarks that CCω preserves the paraconsistency of Cω, Turning now to what CCω lacks, it is interesting to observe that not all the De Morgan laws are provable in this logic. In particular, the thesis $(\neg A \wedge \neg B) \to \neg(A \vee B)$ (i.e., b10 above) is not provable (cf. Proposition A1 in the appendix). Let us label CCω_2 the result of adding b10 to CCω. CCω_2 is paraconsistent in the same sense as CCω. Consequently, we propose to replace CCω with CCω_2. Thus, only sublogics and extensions of CCω with b10 including DHb are considered in the sequel.

All DH-logics defined in the paper are included in G3$_{\mathrm{DH}}$ or in S5$_{\mathrm{DH}}$or in both logics. G3$_{\mathrm{DH}}$ and S5$_{\mathrm{DH}}$, mutually independent logics, are the expansions with DH-negation of negationless 3-valued Gödelian logic G3 and of the 3-valued expansion of negationless Lewis' modal logic S5 (cf. §6 and the appendix). The logics daC, daC′ and PH1 with b10 are some of the DH-logics considered (cf. the appendix). All DH-logics defined in the paper are paraconsistent in the same sense as CCω (cf. Proposition 6.5).

There is a vast literature on dual intuitionistic logic, bi-intuitionistic logic and related topics: cf., for example, [7], [15] and [18] and references therein. Concerning the type of logics here investigated, we have, for example, (a) negation expansions of systems defined with a minimal consequence relation (and without using the conditional): cf., e.g., [4] and [15]; (b) negation expansions of positive systems with a strong conditional: cf., e.g., [4], [18] or [19]. In the present paper, however, dual intuitionistic negation is treated from the double perspective of the Routley-Meyer ternary relational semantics and the De Morgan negation as expressed in the 3-valued logic $\mathrm{G3_{DH}}$, and with an special attention to weak conditionals as Sylvan recommends at the end of his paper (cf. [16], p. 64). The logic $\mathrm{G3_{DH}}$ seems a strong enough dual intuitionistic logic, but it has to be remarked that the methods in the present paper are applicable to logics with some sort of DH-negation not included in $\mathrm{G3_{DH}}$: cf. Remark A2 in the appendix.

As pointed out above, we will provide a Routley-Meyer semantics (RM-semantics) with a set of designated points for extensions of Sylvan and Plumwood's minimal De Morgan logic $\mathrm{B_M}$ extended with the dual intuitionistic negation of the type defined for CCω and included in $\mathrm{G3_{DH}}$ (cf. Remark A2 in the appendix).

RM-semantics was introduced in the early seventies of the past century (cf. [14], [2] and references therein). It was particularly defined for interpreting relevant logics, but it was soon noticed that an ample class of logics not belonging to the relevant logics family could also be characterized by this semantics. RM-semantics is a relational semantics which can essentially be divided in two types: (a) RM-semantics with a set of designated points w.r.t. which validity of formulas is decided; (b) RM-semantics without a set of designated points and where validity of formulas is decided w.r.t. the set of all points. As for RM-semantics with a set of designated points, we have reduced RM-semantics where the set of designated points is reduced to a singleton and unreduced RM-semantics. In what follows, unreduced RM-semantics are defined for the aforementioned extensions of $\mathrm{B_M}$ included in $\mathrm{G3_{DH}}$. We remark that $\mathrm{B_M}$ is the minimal logic interpretable with RM-semantics. Also, we mention that the logics investigated in [10] form a divergent family from the one studied here. In fact, none of those logics —not even $\mathrm{B_{KM}}$, the minimal one— is included in $\mathrm{G3_{DH}}$. Moreover, they are interpreted with an RM-semantics without a set of designated points, not with a reduced RM-semantics or an unreduced one.

The structure of the paper is as follows. In §2, the logic DHb as well as a wealth of its extensions included in $\mathrm{G3_{DH}}$ are defined. In §3, an RM-semantics with a set of designated points is provided for DHb and the extensions of it defined in §2. Weak soundness theorems are proved for all these logics. In §4, we prove some preliminary facts to the proofs of the completeness theorems. In §5, (weak) completeness theorems for all the logics defined in §2 are proved. The section is ended with some remarks on strong completeness. Finally, in §6, we briefly discuss the relations the logics G3, $\mathrm{G3_L}$ and $\mathrm{G3_{DH}}$ maintain to each other ($\mathrm{G3_L}$ is an expansion of $\mathrm{G3_+}$ with a Łukasiewicz type negation —cf. [9]). Also, it is proved that $\mathrm{G3_{DH}}$ and all the logics included in it are paraconsistent. We have added an appendix where some facts stated throughout the paper are proved.

2 The basic logic DHb and its extensions

We begin by defining some basic notions as used in the paper.

Definition 2.1 (Language) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.

Definition 2.2 (Logics) A logic L is a structure $(\mathcal{L}, \vdash_{\mathrm{L}})$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ is a (proof-theoretical) consequence relation defined on $\mathcal{L}$ by a set of axioms and a set of rules of inference. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_{\mathrm{L}} A$ means that A is derivable from the set of wffs Γ in L; and $\vdash_{\mathrm{L}} A$ means that A is a theorem of L).

Definition 2.3 (Extensions and expansions of a propositional logic L) Let $\mathcal{L}$ and $\mathcal{L}'$ be two propositional languages. $\mathcal{L}'$ is a strengthening of $\mathcal{L}$ if the set of wffs of $\mathcal{L}$ is a proper subset of the set of wffs of $\mathcal{L}'$. Next, let L and L$'$ be two logics built upon the propositional languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Moreover, suppose that all axioms of L are theorems of L$'$ and all primitive rules of inference of L are provable in L$'$. Then, L$'$ is an extension of L if $\mathcal{L}$ and $\mathcal{L}'$are the same propositional language; and L$'$ is an expansion of L if $\mathcal{L}'$ is an strengthening of $\mathcal{L}$. An extension L$'$ of L is a proper extension if L is not an extension of L$'$.

The basic logic DHb is the basic extension with a DH-negation of Sylvan and Plumwood's minimal logic $\mathrm{B_M}$ we consider in this paper.

Definition 2.4 (The logic $\mathbf{B_M}$) Sylvan and Plumwood's minimal logic $\mathrm{B_M}$ can be axiomatized with the following axioms and rules of inference (cf. [17]):

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B)$ / $B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

A7. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

A8. $\neg(A \wedge B) \to (\neg A \vee \neg B)$

Rules of inference:

Adjunction (Adj). $A \;\&\; B \Rightarrow A \wedge B$

Modus Ponens (MP). $A \to B \;\&\; A \Rightarrow B$

Suffixing (Suf). $A \to B \Rightarrow (B \to C) \to (A \to C)$

Prefixing (Pref). $B \to C \Rightarrow (A \to B) \to (A \to C)$

Contraposition (Con). $A \to B \Rightarrow \neg B \to \neg A$

Remark 2.5 (The De Morgan laws) The De Morgan laws (T1) $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ and (T2) $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ are provable in $\mathrm{B_M}$ (by A2-A5, A7, A8 and Con; cf. [17]).

Remark 2.6 (The basic logics B_+ and B) Routley and Meyer's basic positive logic B_+ is axiomatized with A1-A6, MP, Adj, Suf and Pref (cf. [14]). In addition, the basic logic B defined by the same authors is axiomatized by adding the double negation axioms ($A \to \neg\neg A$ and $\neg\neg A \to A$) to $\mathrm{B_M}$ (cf. [14]; we note that A7 and A8 are then not independent).

Remark 2.7 (On the double negation axioms) We recall (cf. the introduction) that all extensions of $\mathrm{B_M}$ we investigate in this paper have the double negation elimination axiom, DNE, $\neg\neg A \to A$. Nevertheless, all these extensions lack the double negation introduction axiom, DNI, $A \to \neg\neg A$.

The logic DHb is the basic extension of $\mathrm{B_M}$ with the DH-negation. The basic logic DHb is defined as follows.

Definition 2.8 (The basic logic DHb) The basic logic DHb is axiomatized by adding (A9) $C \to [B \to (A \vee \neg A)]$ and (A10) $C \to [\neg(A \vee \neg A) \to B]$ to $\mathrm{B_M}$.

We note that A9 and A10 are theorems of $\mathrm{CC}\omega$ but that A10 is not provable in $\mathrm{C}\omega$. On the other hand, it has to be remarked that $B \to (A \vee \neg A)$ is not sufficient for axiomatizing DH-negation in certain logics (cf. Proposition 2.11 below and §4). Also, we note the ensuing proposition.

Proposition 2.9 (Some theorems of DHb) *The following are provable in DHb:*

$$\textit{T3 (PEM).}\ A \vee \neg A$$
$$\textit{T4 (Principle of Non Contradiction —PNC).}\ \neg(A \wedge \neg A)$$
$$\textit{T5 (Restricted ECQ —rECQ).}\ (\neg A \wedge \neg\neg A) \to B$$
$$\textit{T6 (DNE).}\ \neg\neg A \to A$$

Proof T3 is immediate by A9; T4, by T2 and T3 in the form $\neg A \vee \neg\neg A$; T5, by A10 and T1; finally, T6 is proved as follows: we have (1) $\neg\neg A \to (A \vee \neg A)$, by A9, and (2) $(\neg A \wedge \neg\neg A) \to A$ by T5. Now we use (3) $(\neg\neg A \wedge \neg\neg A) \to [(\neg\neg A \wedge A) \vee (\neg\neg A \wedge \neg A)]$ and (4) $[(\neg\neg A \wedge A) \vee (\neg\neg A \wedge \neg A)] \to A$, which are immediately provable by B_+ and 1 and 2, respectively. Finally, we get (5) $(\neg\neg A \wedge \neg\neg A) \to A$ by 3 and 4, and then (6) $\neg\neg A \to A$ by 5 and B_+.

There are other remarkable facts concerning axioms A9 and A10. In Proposition 2.10 it is proved that they are independent within the context of Routley and Meyer's basic logic B. In Proposition 2.11, it is shown that A9 and A10 are not derivable from Anderson and Belnap's *logic of entailment* E plus the conditioned PEM, $B \to (A \vee \neg A)$. Finally, in Proposition 2.12, it is proved that A9 and A10 are theorems of DW if the *Assertion* axiom and the conditioned PEM are added. By a1, a2, ..., a44, we refer to the items in Lemma 2.13 below.

Consider the following truth-tables. There are three tables for $\to$ (designated values are starred).

t1

$\to$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
*2	0	0	2	2
*3	0	0	0	2

t2

$\to$	0	1	2	3
0	2	2	2	3
1	0	2	0	3
*2	0	0	2	3
*3	0	0	0	3

t3

$\to$	0	1	2	3
0	2	2	2	2
1	0	2	0	2
*2	0	0	2	2
*3	0	0	0	2

The tables for $\wedge$, $\vee$ and $\neg$ are the following:

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

	$\neg$
0	3
1	2
*2	1
*3	0

Let p_i, p_m and p_r be distinct propositional variables in the propositions to follow. We have:

Proposition 2.10 (B, A9 and A10) *Axioms A9 and A10 are independent, given Routley and Meyer's basic logic B. That is, (1) A10 is not derivable from B and A9; (2) A9 is not derivable from B and A10.*

Proof (1) Table t2 together with tables for $\wedge$, $\vee$ and $\neg$ verifies all axioms and rules of B plus A9 but falsifies $p_i \to [\neg(p_m \vee \neg p_m) \to p_r]$ for any assignment v such that $v(p_i) = v(p_m) = v(p_r) = 1$.

(2) Table t1 together with tables for $\wedge$, $\vee$ and $\neg$ verifies all axioms and rules of B plus A10 but falsifies $p_i \to [p_m \to (p_r \vee \neg p_r)]$ for any assignment v such that $v(p_i) = v(p_m) = v(p_r) = 3$.

Proposition 2.11 (E, A9 and A10) *A9 and A10 are not derivable from E plus the CPEM axiom $B \to (A \vee \neg A)$. (E is Anderson and Belnap's logic of entailment and can be axiomatized by adding to B a2, a3, a5, a10, a34 and the contraposition axiom $(A \to B) \to (\neg B \to \neg A)$ —notice that the rule Con is then not independent; cf. [1].)*

Proof Table t3 together with tables for for $\wedge$, $\vee$ and $\neg$ verifies all axioms and rules of E plus $B \to (A \vee \neg A)$ but falsifies $p_i \to [p_m \to (p_r \vee \neg p_r)]$ and $p_i \to [\neg(p_m \vee \neg p_m) \to p_r]$ for any assignment v such that $v(p_i) = v(p_m) = v(p_r) = 1$.

Proposition 2.12 (DW, A9 and A10) *A9 and A10 are derivable from DW plus CPEM, $B \to (A \vee \neg A)$, and the Assertion axiom a14, $A \to [(A \to B) \to B]$. (DW is axiomatized by adding to B the contraposition axiom $(A \to B) \to (\neg B \to \neg A)$ —notice that the rule Con is then not independent; cf. [14])*

Proof (1) A10 is provable by $\neg(A \vee \neg A) \to (C \to B)$ and a14 in the form $C \to [(C \to B) \to B]$; (2) A9 is provable by A10, the contraposition axiom and the double negation axioms.

Finally, we will define the set of extensions of DHb investigated in the present paper. We shall consider the extensions of DHb built by the axioms and rules in the following Lemma 2.13. These axioms and rules are provable in $G3_{DH}$, the expansion of the positive fragment of Gödelian 3-valued logic G3 with DH-negation.

Important logics can be axiomatized by using this set of axioms and rules as, for example, the extensions of $C\omega$ defined in the Appendix (notice that any subset of the theses a1 through a21 and a25 through a40 is a subset of the set of all theorems of $CC\omega$).

Lemma 2.13 (A set of theses and rules of $G3_{DH}$) *The following theses and rules are provable in $G3_{DH}$:*

a1. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
a2. $(B \to C) \to [(A \to B) \to (A \to C)]$
a3. $(A \to B) \to [(B \to C) \to (A \to C)]$
a4. $[A \wedge (A \to B)] \to B$
a5. $[A \to (A \to B)] \to (A \to B)$
a6. $A \to [[A \to (A \to B)] \to B]$

a7. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
a8. $(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$
a9. $[A \to (B \to C)] \to [(A \wedge B) \to C]$
a10. $[[(A \to A) \wedge (B \to B)] \to C] \to C$
a11. $A \Rightarrow (A \to B) \to B$
a12. $A \to [[A \to (B \to C)] \to (B \to C)]$
a13. $[A \to [B \to (C \to D)]] \to [B \to [A \to (C \to D)]]$
a14. $A \to [(A \to B) \to B]$
a15. $[A \to (B \to C)] \to [B \to (A \to C)]$
a16. $(A \wedge B) \to [[A \to (B \to C)] \to C]$
a17. $(A \to B) \to [[A \wedge (B \to C)] \to C]$
a18. $[A \wedge (B \to C)] \to [(A \to B) \to C)]$
a19. $B \to [[A \to (B \to C)] \to (A \to C)]$
a20. $A \to (A \to A)$
a21. $A \to [B \to (A \vee B)]$
a22. $(A \to B) \vee (B \to A)$
a23. $[A \to (B \vee C)] \to [(A \to B) \vee (A \to C)]$
a24. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$
a25. $B \to (A \to A)$
a26. $(A \to B) \to [C \to (A \to B)]$
a27. $A \to (B \to A)$
a28. $A \to [B \to (C \to A)]$
a29. $(A \vee B) \to [(A \to B) \to B]$
a30. $A \to [B \to (A \wedge B)]$
a31. $[(A \wedge B) \to C] \to [A \to (B \to C)]$
a32. $\neg B \to [\neg A \vee \neg(A \to B)]$
a33. $A \to [B \vee \neg(A \to B)]$
a34. $(A \to B) \to (\neg A \vee B)$
a35. $(A \vee \neg B) \vee (A \to B)$
a36. $(\neg A \wedge B) \to (A \to B)$
a37. $\neg(A \to B) \to (A \vee \neg B)$
a38. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$
a39. $\neg A \to (B \to \neg A)$
a40. $\neg(A \to B) \to \neg B$
a41. $\neg(A \to B) \to (B \to A)$
a42. $\neg\neg A \to (\neg A \to B)$
a43. $A \vee \neg B \Rightarrow \neg A \to \neg B$
a44. $A \vee B \Rightarrow \neg A \to B$

Proof Immediate by using the matrix MG3$_{\mathrm{DH}}$ in Definition 6.2.

Remark 2.14 (On DH-negation extensions of B) Notice that Routley and Meyer's basic logic B (cf. Remark 2.6) cannot be extended with DH-negation on pain of collapse into Boolean-negation: by using the DNI axiom, the ECQ axiom is immediately derivable and DH-negation collapses into Boolean-negation.

Remark 2.15 (Theses and rules not provable in DH-logics) By using the matrices MG3$_{\mathrm{DH}}$ and MS5$_{\mathrm{DH}}$ (cf. Definitions 6.2 and A2), it is immediately proved that the following theses and rules mentioned in the introduction to the paper are not provable in the DH-logics: b1, b2, b3, b4, b5, b7, b13 and b14.

3 RM-semantics for the DH-logics

As pointed out above, in what follows, by an DH-logic we mean an extension of the basic logic DHb with some subset of the axioms and rules a1 through a44. We begin by defining DH-models, models for extensions of DHb, together with the accompanying definitions of truth and validity.

Definition 3.1 (DH-models) A DH-model, M, is a structure with at least the following items: (a) A set K and a subset of it, O. (b) A ternary relation R and a unary operation $*$ defined on K subject at least to the following definitions and postulates for all $a, b, c, d \in K$:

$$
\begin{aligned}
\text{d1. } & a \leq b =_{\mathrm{df}} \exists x \in O Rxab \\
\text{d1}'. \ & a = b =_{\mathrm{df}} a \leq b \ \ \& \ \ b \leq a \\
\text{d2. } & R^2abcd =_{\mathrm{df}} \exists x \in K(Rabx \ \ \& \ \ Rxcd) \\
\text{P1. } & a \leq a \\
\text{P2a. } & (a \leq b \ \ \& \ \ Rbcd) \Rightarrow Racd \\
\text{P2b. } & (a \leq b \ \ \& \ \ b \leq c) \Rightarrow a \leq c \\
\text{P2c. } & (d \leq b \ \ \& \ \ Rabc) \Rightarrow Radc \\
\text{P2d. } & (c \leq d \ \ \& \ \ Rabc) \Rightarrow Rabd \\
\text{P3. } & a \leq b \Rightarrow b^* \leq a^* \\
\text{P4. } & a^* \leq a
\end{aligned}
$$

(c) A valuation relation $\vDash$ from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

$$
\begin{aligned}
\text{(i). } & (a \leq b \ \ \& \ \ a \vDash p) \Rightarrow b \vDash p \\
\text{(ii). } & a \vDash A \wedge B \text{ iff } a \vDash A \text{ and } a \vDash B \\
\text{(iii). } & a \vDash A \vee B \text{ iff } a \vDash A \text{ or } a \vDash B \\
\text{(iv). } & a \vDash A \to B \text{ iff for all } b, c \in K, (Rabc \ \ \& \ \ b \vDash A) \Rightarrow c \vDash B \\
\text{(v). } & a \vDash \neg A \text{ iff } a^* \nvDash A
\end{aligned}
$$

Semantical postulates $\mathrm{Pj}_1, ..., \mathrm{Pj}_n$ can be appended to M as additional elements.

Structures of the form $(O, K, R, *, \vDash)$ satisfying d1, d1′, d2, P1, P2a, P2b, P2c, P2d, P3, P4 and clauses (i), (ii), (iii), (iv) and (v) are the basic structures and in fact characterize the logic DHb (they are labelled DHb-models). Introduction of additional postulates serve to determine extensions and expansions of DHb interpretable in unreduced RM-semantics.

Definition 3.2 (Truth) Let a class of DH-models $\mathcal{M}$ be defined and $\mathrm{M} \in \mathcal{M}$. A wff A is true in M (in symbols, $\vDash_{\mathrm{M}} A$) iff $a \vDash A$ for all $a \in O$.

Definition 3.3 (Validity) Let a class of DH-models $\mathcal{M}$ be defined. A wff A is valid in $\mathcal{M}$ (in symbols, $\vDash_{\mathcal{M}} A$) iff A is true in every $\mathrm{M} \in \mathcal{M}$.

The following lemmas, Lemma 3.4 and Lemma 3.6, and Proposition 3.6 are useful for proving that the axioms of DHb are valid and that its rules of inference preserve validity in any DH-model. Then, (weak) soundness of DHb is immediate.

Lemma 3.4 (Hereditary Lemma) *For any DH-model, $a, b \in K$ and wff A, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.*

Proof Induction on the length of A. The conditional case is proved with P2a and the negation case is proved with P3.

Lemma 3.5 (Entailment Lemma) *Let a class of DH-models $\mathcal{M}$ be defined. For any wffs A, B, $\vDash_{\mathcal{M}} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$ for all $a \in K)$ in all M $\in \mathcal{M}$.*

Proof From left to right, by P1; from right to left, by Lemma 3.4.

Proposition 3.6 (A couple of postulates) *Let a class of DH-models $\mathcal{M}$ be defined. Then, the following semantical postulates P4a and P4b are provable in any $M \in \mathcal{M}$: (P4a) $a^{**} \leq a^*$; (P4b) $a^{**} \leq a$.*

Proof P4a is immediate by P4; P4b follows immediately by P2b, P4 and P4a.

Let L be a DH-logic and L-models be defined. Below, it is proved that all theorems of DHb are L-valid. Then, soundness of DHb is a corollary of this fact.

Proposition 3.7 (All theorems of DHb are DH-valid) *For any wff A, if $\vdash_{DHb} A$, then A is DH-valid (i.e., valid in any class of DH-models).*

Proof The proof for the axioms and rules of $\mathrm{B_M}$ can be found in [14], Chapter 4. So, let us prove the DH-validity of A9 and A10. We suppose that we are given a class of DH-models $\mathcal{M}$ and some $\mathrm{M} \in \mathcal{M}$. Then, we prove that A9 and A10 are true in M. We lean upon the Entailment and Hereditary Lemmas, Lemmas 3.4 and 3.5, respectively. By i, ii, etc., we refer to clauses (i), (ii), etc. in Definition 3.1.

(a) *A9,* $C \to [B \to (A\vee\neg A)]$*, is true in M*: For reductio, suppose that there are wffs A, B, C and $a \in K$ in M such that (1) $a \vDash C$ but (2) $a \nvDash B \to (A\vee\neg A)$. Then, we have $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash B$, (5) $c \nvDash A$ and (6) $c \nvDash \neg A$. By 6 and v, (7) $c^* \vDash A$ follows. But 7 contradicts 5 by applying P4 and Lemma 3.4 to 7. (We note that P4 is used in [14] to validate $B \to (A \vee \neg A)$, a weaker version of A9.)

(b) *A10,* $C \to [\neg(A \vee \neg A) \to B]$*, is true in M*: For reductio, suppose that there are wffs A, B, C and $a \in K$ in M such that (1) $a \vDash C$ but (2) $a \nvDash \neg(A \vee \neg A) \to B$. By 2 and iv, we have $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash \neg(A \vee \neg A)$, (5) $c \nvDash B$. By 4, v and iii, we get (6) $b^* \nvDash A$ and (7) $b^* \nvDash \neg A$, whence by v, we have (8) $b^{**} \vDash A$. But by Lemma 3.4, P4a and 8, (9) $b^* \vDash A$ is derivable, contradicting 6.

Corollary 3.8 (Soundness of DHb) *For any wff A, if* $\vdash_{DHb} A$*, then* $\vDash_{DHb} A$.

Proof Immediate by Proposition 3.7, since a DH-model is a DH-model.

In what follows, we proceed to the soundness proofs of the DH-logics. The basic notion is "corresponding postulate" (cf. [14], Chapter 4). We give a corresponding postulate to each one of the axioms and rules a1 through a44. Then, Lemma 3.10 shows that, given a class of DH-models $\mathcal{M}$ and a DH-model M such that $\mathrm{M} \in \mathcal{M}$, a thesis or rule a_k ($1 \leq k \leq 44$) is true in M (or preserves truth in M, as the case may be) provided its corresponding semantic postulate pa_k holds in M. Next, DH-models for DH-logics are simply defined by adding to DHb-models the postulates corresponding to the axioms or rules added to the logic DHb in order to define the extension of DHb in question. Then, soundness of each one of the DH-logics is immediate by leaning upon soundness of DHb and Lemma 3.10.

Definition 3.9 (Postulates corresponding to a1-a44) Below, we provide postulates corresponding to each one of the items a1-a44 in Lemma 2.13.

pa1. $Rabc \Rightarrow \exists x(Rabx \;\&\; Raxc)$

pa2. $R^2abcd \Rightarrow \exists x(Rbcx \;\&\; Raxd)$

pa3. $R^2abcd \Rightarrow \exists x(Racx \;\&\; Rbxd)$

pa4. $Raaa$

pa5. $Rabc \Rightarrow R^2abbc$

pa6. $Rabc \Rightarrow R^2baac$

pa7. $R^2abcd \Rightarrow \exists x, y(Racx \;\&\; Rbcy \;\&\; Rxyd)$

pa8. $R^2abcd \Rightarrow \exists x, y(Racx \;\&\; Rbcy \;\&\; Ryxd)$

pa9. $Rabc \Rightarrow R^2abbc$

pa10. $\exists x \in Z\ Raxa$ [Za iff for all $b, c \in K, Rabc \Rightarrow \exists x \in O\ Rxbc$]

pa11. $\exists x \in O\ Raxa$

pa12. $R^2abcd \Rightarrow R^2bacd$

pa13. $R^3abcde \Rightarrow R^3acbde$

pa14. $Rabc \Rightarrow Rbac$

pa15. $R^2abcd \Rightarrow R^2acbd$

pa16. $Rabc \Rightarrow R^2baac$

pa17. $Rabc \Rightarrow \exists x(Rabx\ \ \&\ \ Rbxc)$

pa18. $Rabc \Rightarrow \exists x(Rbax\ \ \&\ \ Raxc)$

pa19. $R^2abcd \Rightarrow R^2bcad$

pa20. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$

pa21. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$

pa22. $(Rabc\ \ \&\ \ Rade\ \ \&\ \ a \in O) \Rightarrow (b \leq e \text{ or } d \leq c)$

pa23. $(Rabc\ \ \&\ \ Rade) \Rightarrow \exists x[(Rabx \text{ or } Radx)\ \ \&\ \ x \leq c\ \ \&\ \ x \leq e]$

pa24. $(Rabc\ \ \&\ \ Rade) \Rightarrow \exists x[(Raxc \text{ or } Raxe)\ \ \&\ \ b \leq x\ \ \&\ \ d \leq x]$

pa25. $Rabc \Rightarrow b \leq c$

pa26. $R^2abcd \Rightarrow Racd$

pa27. $Rabc \Rightarrow a \leq c$

pa28. $R^2abcd \Rightarrow a \leq d$

pa29. $Rabc \Rightarrow (Rbac\ \ \&\ \ a \leq c)$

pa30. $Rabc \Rightarrow (a \leq c\ \ \&\ \ b \leq c)$

pa31. $R^2abcd \Rightarrow \exists x(Raxd\ \ \&\ \ b \leq x\ \ \&\ \ c \leq x)$

pa32. $Ra^*a^*a^*$

pa33. Ra^*aa

pa34. Raa^*a

pa35. $Rabc \Rightarrow (b \leq a \text{ or } a \leq c^*)$

pa36. $Rabc \Rightarrow (b \leq a^* \text{ or } a \leq c)$

pa37. $Ra^*bc \Rightarrow (b \leq a \text{ or } a^* \leq c)$

pa38. $Ra^*bc \Rightarrow (b \leq a^* \text{ or } a \leq c)$

pa39. $Rabc \Rightarrow c^* \leq a^*$

pa40. $Ra^*bc \Rightarrow a^* \leq c$

pa41. $(Rabc\ \ \&\ \ Ra^*de) \Rightarrow (d \leq c \text{ or } b \leq e)$

pa42. $Rabc \Rightarrow a^{**} \leq b^*$

pa43. $\exists x \in O(x \leq a^*\ \ \&\ \ a^* \leq x^*)$

pa44. $\exists x \in O(x \leq a^*\ \ \&\ \ x \leq a)$

Lemma 3.10 (DH-validity of a1-a44) *Let $\mathcal{M}$ be a class of DH-models and $M \in \mathcal{M}$. Then, for any k ($1 \leq k \leq 10$; or $12 \leq k \leq 42$) ak is true in M if pak* *holds in M; and for any k ($k \in \{11, 43, 44\}$), a_k preserves truth in M if pa_k holds in M.*

Proof The proof of the validity of a1-a31 can be found in [12]. The proof of a32-44 is similar to that given in [14], Chapter 4, for extensions of Routley and Meyer's basic positive logic B. Let us prove some cases:

(a) *a36, $(\neg A \wedge B) \to (A \to B)$, is true in M*: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \vDash \neg A \wedge B$ but (2) $a \nvDash A \to B$. By 2 and iv, there are $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash A$ and (5) $c \nvDash B$. On the other hand, by 1, ii and v, we have (6) $a^* \nvDash A$ and (7) $a \vDash B$. Then, by 3 and pa36 we have (8) $b \leq a^*$ or (9) $a \leq c$. But by 4, 8 and Lemma 3.4 we have (10) $a^* \vDash A$, contradicting 6; and by 7, 9 and Lemma 3.4, we get (11) $c \vDash B$, contradicting 5.

(b) *a40, $\neg(A \to B) \to \neg B$, is true in M*: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \vDash \neg(A \to B)$ but (2) $a \nvDash \neg B$, i.e., (3) $a^* \vDash B$, by v. By 1 and v, we have (4) $a^* \nvDash A \to B$, whence, by iv, there are $b, c \in K$ in M such that (5) Ra^*bc, (6) $b \vDash A$ and (7) $c \nvDash B$. Then, by 5 and pa40, we have (8) $a^* \leq c$, whence by 3 and Lemma 3.4, we get $c \vDash B$, contradicting 7.

(c) *a44, $A \vee B \Rightarrow \neg A \to B$, preserves truth in M*: For reductio suppose that there are wffs A, B such that (1) $\vDash_{\mathrm{M}} A \vee B$ but (2) $\nvDash_{\mathrm{M}} \neg A \to B$. By Lemma 3.5, there is some $a \in K$ such that (3) $a \vDash \neg A$ (i.e., $a^* \nvDash A$) and (4) $a \nvDash B$. But, by pa44 we have some $x \in O$ such that (5) $x \leq a^*$ and (6) $x \leq a$. Given 1, we get (7) $x \vDash A$ or $x \vDash B$, whence by 5, 6 and Lemma 3.4, we obtain (8) $a^* \vDash A$ or $a \vDash B$, contradicting 3 and 4.

Definition 3.11 (L-models) Let L be a DH-logic. An L-model is defined when adding to DHb-models the semantical postulates corresponding to the axioms added to DHb for axiomatizing L. For example, consider the extension of DHb axiomatized by a3, a5 and a14. Then, a DH-model for this system is a structure $(O, K, R, *, \vDash)$ where O, K, R, $*$, and $\vDash$ are defined exactly as in Definition 3.1, save for the addition of the postulates pa3, pa5 and pa14. (The notion of L-validity is defined according to the general Definition 3.3. Notice that the system just defined is the expansion of positive relevance logic R (cf. [1]) with the basic dual intuitionistic negation defined above).

Theorem 3.12 (Soundness of DH-logics) *Let L be a DH-logic. For any wff A, if $\vdash_L A$, then $\vDash_L A$.*

Proof By Proposition 3.7 and Lemma 3.10, given Definition 3.11.

Concerning L-models, we note the following remark (cf. Remark 2.14).

Remark 3.13 (Bb-models) We note that Bb-models, RM-models for Boolean-negation, are defined by adding the postulate $a \leq a^*$ to DHb-models. Bb-models characterize DHb plus the axioms $C \to [B \to \neg(A \wedge \neg A)]$ (A9′) and $C \to [(A \wedge \neg A) \to B]$ (A10′).

4 Completeness. Preliminary notions and lemmas

Firstly, we define some preliminary notions and prove some facts necessary to show the completeness of the DH-logics w.r.t. the RM-semantics defined in Section 3. We begin by defining the notion of a DH-theory and the classes of DH-theories of interest in the paper. Then, a couple of facts about DH-theories are proved and the notion of a canonical DH-model is stated.

Definition 4.1 (DH-theories) Let L be a DH-logic. An L-theory is a set of wffs closed under Adjunction (Adj) and L-entailment (L-ent). That is, a is an L-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of L and $A \in a$, then $B \in a$.

By the term DH-theory, we will generally refer to any theory defined upon a DH-logic as just indicated. The classes of DH-theories of interest in the present paper are remarked in the following definition.

Definition 4.2 (Classes of DH-theories) Let L be a DH-logic and a an L-theory. We set: (1) a is *prime* iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$. (2) a is *empty* iff it contains no wffs. (3) a is *regular* iff a contains all theorems of L. (4) a is *trivial* iff every wff belongs to it. (5) a is *a-consistent* (consistent in an absolute sense) iff a is not trivial.

Proposition 4.3 (On non-empty DH-theories) *Let L be a DH-logic and a a non-empty L-theory. Then, $A \vee \neg A \in a$.*

Proof It is immediate by A9.

Proposition 4.4 (On a-inconsistent DH-theories) *Let a be a DH-logic and a an L-theory. Then, a is a-inconsistent iff $\neg(A \vee \neg A) \in a$ for some wff A.*

Proof From left to right, it is obvious; the inverse direction follows immediately by A10.

Nevertheless, notice that a may be a-consistent while still containing a contradiction (cf. Proposition 6.5 below).

Definition 4.5 (Main notions for defining canonical models) Let L be a DH-logic and K^T be the set of all L-theories. Then, the ternary relation R^T is defined in K^T as follows: for any $a, b, c \in K^T$, $R^T abc$ iff for any wffs A, B, $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Next, let K^C be the set of all prime, non-empty and a-consistent L-theories, O^C be the subset of K^C consisting of all regular L-theories and R^C be the restriction of R^T to K^C. On the other hand, let $*^C$be defined on K^C as follows: for all $a \in K^C$, $a^{*^C} = \{A \mid \neg A \notin a\}$. Finally, the relation $\vDash^C$is defined as follows: for each formula A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$.

Definition 4.6 (Canonical DH-models) Let L be a DH-logic. The structure $(O^C, K^C, R^C, *^C, \vDash^C)$, where $O^C, K^C, R^C, *^C$ and $\vDash^C$ are defined as in Definition 4.5 above, is the canonical L-model.

Next, a series of Lemmas follows. These lemmas will be used in the proof that canonical L-models are indeed L-models.

In the proofs that follow, we suppose that we are given a DH-logic L: some of the following lemmas are not provable for weaker logics. We remark that A9 and A10 are needed in the proof of Lemmas 4.7, 4.8 and 4.15 (these lemmas are not provable with weaker versions of A9 and A10). Also, we use the label $\mathrm{L_{TH}}$ to refer to all theorems of L. It is obvious that $\mathrm{L_{TH}}$ is a (regular) theory.

Lemma 4.7 (Defining x for a, b in R^T) *Let a, b be non-empty L-theories. The set $x = \{B \mid \exists A(A \to B \in a \ \ \& \ \ A \in b)\}$ is a non-empty L-theory such that $R^T abx$.*

Proof It is easy to show that x is a L-theory. Next, $R^T abx$ is immediate by definition of R^T. Finally, x is non-empty: let $A \in a$, $B \in b$. By A9 and $R^T abx$, $A \vee \neg A \in x$ (notice that CPEM, $B \to (A \vee \neg A)$, is not sufficient).

Lemma 4.8 (Extending b in $R^T abc$ to x in K^C) *Let a, b be non-empty L-theories and c a prime and a-consistent L-theory such that $R^T abc$. Then, there is a prime, a-consistent (and non-empty) L-theory x such that $b \subseteq x$ and $R^T axc$.*

Proof By using the Extension Lemma or Kuratowski-Zorn's Lemma, b is extended to a prime L-theory x such that $b \subseteq x$ and $R^T axc$ (cf. [14], pp. 309, ff). Next, it is shown that x is a-consistent. Let $A \in a$, B be an arbitrary wff and suppose, for reductio, that x is a-inconsistent. Then $\neg(C \vee \neg C) \in x$, by Proposition 4.4. By A10, $A \to [\neg(C \vee \neg C) \to B]$ is an L-theorem. So, $\neg(C \vee \neg C) \to B \in a$, whence $B \in c$, by $R^T axc$, contradicting the a-consisteny of c (notice that the simpler $\neg(A \vee \neg A) \to B$ is not sufficient).

Lemma 4.9 (Extending a in $R^T abc$ to x in K^C) *Let a, b be non-empty L-theories and c be a prime, a-consistent L-theory such that $R^T abc$. Then, there is a prime, a-consistent (and non-empty) L-theory x such that $a \subseteq x$ and $R^T xbc$.*

Proof As in the previous lemma, it is shown that there is a prime theory x such that $a \subseteq x$ and $R^T xbc$. Next, it is shown that x is a-consistent. Suppose it is not and let $A \in b$ and B be an arbitrary wff. As x is supposed to be trivial, $A \to B \in x$. Then, $B \in c$ ($R^T xbc$, $A \to B \in x$, $A \in b$ and definition of R^T) contradicting the a-consistency of c.

Consider now the following definition.

Definition 4.10 (The relation $\leq^C$) For any $a, b \in K^C$, $a \leq^C b$ iff $\exists x \in O^C$ $R^C xab$.

The following lemma shows that the relation $\leq^C$ is just set inclusion between a-consistent, non-empty and prime L-theories.

Lemma 4.11 **($\leq^C$ and $\subseteq$ are coextensive)** *For any $a, b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof From left to right, it is immediate by using A1 of $\mathrm{B_M}$. Suppose now $a \subseteq b$ for $a, b \in K^C$. Clearly $R^T \mathrm{L_{TH}} aa$ (cf. Definitions 4.1 and 4.5). By Lemma 4.9, $\mathrm{L_{TH}}$ is extended to a member x in K^C such that $R^C xaa$, By the hypothesis $R^C xab$, i.e., $a \leq^C b$ by Definition 4.10, since $x \in O^C$.

Lemma 4.12 **(Extension to prime L-theories)** *Let a be an L-theory and A a wff such that $A \notin a$. Then, there is a prime L-theory x such that $a \subseteq x$ and $A \notin x$.*

Proof By direct application of Kuratowski-Zorn's Lemma as in [14], Chapter 4, pp. 310-311.

In what follows, we investigate the operation $*$.

Lemma 4.13 **(Primeness of $*$-images)** *Let a be a prime L-theory. Then, a^{*^C} is a prime theory as well.*

Proof As there is no danger of confusion between a^* in K and the canonical L-theory a^{*^C} in K^C, we omit the supersript ^{C} above $*$ in this and the proofs to follow. Then, a^* is closed under L-ent by Con; a^* is closed under Adj by T2 and, finally, a^* is prime by T1.

Lemma 4.14 **($*^C$ is an operation on K^C)** *Let a be a prime, non-empty and a-consistent L-theory. Then, a^{*^C} is a prime, non-empty and a-consistent L-theory as well.*

Proof By Lemma 4.13, a^* is a prime L-theory. Next, it is shown that if a is non-empty and a-consistent, then a^* is also non-empty and a-consistent. (a) a^* is non-empty. If a^* is empty, then $A \vee \neg A \notin a^*$, whence $\neg(A \vee \neg A) \in a$, contradicting the a-consistency of a (cf. Proposition 4.4). (b) a^* is a-consistent. As a is not trivial, we have $A \notin a$ for some wff A. As a is not empty, $A \vee \neg A \in a$ by Proposition 4.3. Then, $A \in a$ or $\neg A \in a$ by primeness of a, whence $\neg A \in a$ and, thus, $A \notin a^*$ showing the a-consistency of a.

Lemma 4.15 **($\vDash^C$ and clauses (i)-(v))** *For any $a, b, c \in K^C$ and wffs A, B: (i) $(a \leq^C b \;\;\&\;\; a \vDash^C p) \Rightarrow b \vDash^C p$; (ii) $a \vDash^C A \wedge B$ iff $a \vDash^C A$ and $a \vDash^C B$; (iii) $a \vDash^C A \vee B$ iff $a \vDash^C A$ or $a \vDash^C B$; (iv) $a \vDash^C A \to B$ iff for all $b, c \in K^C$, $(R^C abc \;\;\&\;\; b \vDash^C A) \Rightarrow c \vDash^C B$; (v) $a \vDash^C \neg A$ iff $a^{*^C} \nvDash^C A$.*

Proof Similar to those in relevant logics, save that Lemmas 4.7, 4.8 and 4.9 are used to prove non-emptiness and a-consistency when required (cf. [14], Chapter 4).

5 Completeness of the DH-logics

Let L be a DH-logic. Completeness w.r.t. the semantics defined in §3 is proved by a canonical model construction. Once the canonical L-model is shown an L-model, it can be proved that if A is not an L-theorem, then A fails to belong to some $a \in O^C$ in the canonical L-model, whence it is immediate that A is not L-valid. Given Lemmas 4.14 and 4.15, in order to prove that the canonical L-model is an L-model, we need to prove the following two facts (1) the set O^C is non-empty; (2) the postulates are canonically valid.

Corollary 5.1 **(O^C is not empty)** *Let L be a DH-logic and* $(O^C, K^C, R^C, *^C, \vDash^C)$ *be the canonical L-model. Then, the set* O^C *is not empty.*

Proof Clearly, $\mathrm{L_{TH}}$ is a-consistent since all axioms and rules of L are axioms and rules of classical propositional logic when read with the classical connectives. Then, Corollary 5.1 is immediate by Lemma 4.12.

Next, let us prove that the postulates are canonically valid. We recall that a canonical L-model is a structure $(O^C, K^C, R^C, *^C, \vDash^C)$, where $O^C, K^C, R^C, *^C$, and $\vDash^C$ are defined as indicated in Definition 4.6.

Lemma 5.2 **(The postulates are canonically valid)** *Let L be a DH-logic. Then, (1) P1, P2a, P2b, P2c, P2d, P3 and P4 hold in all canonical DH-models. (2) pak holds in the canonical L-model if ak is provable in L* $(1 \leq k \leq 44)$.

Proof The proof is similar to that provided in [14], Chapter 4, for extensions of Routley and Meyer's basic logic B. Actually, a proof for P1, P2a, P2b, P2c, P2d and P3 can be found in the aforementioned chapter and P4 is proved below. Then, pa$k$ holds in the canonical L-model if ak is provable in L $(1 \leq k \leq 44)$. Now, concerning pa1-pa31, the proof can be found in [12]. And concerning pa32-pa44, we prove the canonical validity of the postulates used in Lemma 3.10, as a way of an example.

(a) *P4,* $a^* \leq a$*, is provable in the canonical L-model*: Suppose $a \in K^C$ and (1) $A \in a^*$. We have to prove $A \in a$. By 1, we have (2) $\neg A \notin a$ but (3) $A \vee \neg A \in a$ follows by Proposition 4.3, so we get (4) $A \in a$ by primeness of a.

(b) *pa36,* $Rabc \Rightarrow (b \leq a^*$ or $a \leq c)$*, is provable in the canonical L-model:* Let $a, b, c \in K^C$ and suppose (1) $R^C abc$ and, for reductio, (2) $A \in b$, (3) $A \notin a^*$, i.e., $\neg A \in a$, (4) $B \in a$ and (5) $B \notin c$ for wffs A, B. By 3 and 4, we have (6) $\neg A \wedge B \in a$ and by a36 we get (7) $\vdash_{\mathrm{L}} (\neg A \wedge B) \to (A \to B)$. Thus, by 6 and 7, (8) $A \to B \in a$ is derivable. Finally, by 1, 2 and 8, we obtain (9) $B \in c$. But 5 and 9 contradict each other.

(c) *pa40,* $Ra^*bc \Rightarrow a^* \leq c$ *is provable in the canonical L-model*: Let $a, b, c \in K^C$ and suppose (1) $R^C a^* bc$ and, for reductio, (2) $A \in a^*$ and (3) $A \notin c$ for some wff A (cf. Lemma 4.11). Let (4) $B \in b$ (b is non-empty). By 1, 3 and 4, we get (5) $B \to A \notin a^*$ (cf. Definition 4.5), whence (6) $\neg(B \to A) \in a$ is derivable. Now, by a40, we have (7) $\vdash_{\mathrm{L}} \neg(B \to A) \to \neg A$. Thus, we get (8) $\neg A \in a$, i.e, $A \notin a^*$, contradicting 2.

(d) pa44, $\exists x \in O(x \leq a^* \ \& \ x \leq a)$ *is provable in the canonical L-model*: Let (1) $a \in K^C$ and define (2) $z = \{A \mid \vdash_{\mathrm{L}} A\}$. We prove (3) $R^T zza^*$. Suppose, for some wffs A, B, (4) $A \to B \in z$ and $A \in z$. Then, we have (5) $\vdash_{\mathrm{L}} A \to B$ and $\vdash_{\mathrm{L}} A$ and so (6) $\vdash_{\mathrm{L}} B$. Suppose now (7) $C \notin a$ (a is a-consistent). Then, we have (8) $\vdash_{\mathrm{L}} B \vee C$, whence by a44, we get (9) $\vdash_{\mathrm{L}} \neg B \to C$. Next, by 7 and 9 we obtain (10) $\neg B \notin a$, i.e., (11) $B \in a^*$, as it was to be proved. It remains to extend z to prime (regular) and a-consistent theories y, x such that $R^C yxa^*$, i.e., $x \leq^C a^*$ (we use Lemmas 4.8 and 4.9). Then, $R^C yxa$ is proved by P4, P2d, Lemma 4.11 and $R^C yxa^*$.

Proposition 5.3 (The canonical model is a model) *Let L be a DH-logic. The canonical L-model is indeed an L-model.*

Proof Given Definition 4.6 and Corollary 5.1, the proof follows by Lemma 4.14 ($*^C$is a operation on K^C), Lemma 4.15 (Adequacy of the canonical clauses) and Lemma 5.2. (The postulates hold canonically).

Theorem 5.4 (Completeness of the DH-logics) *Let L be a DH-logic. For any wff A, if* $\vDash_L A$, *then* $\vdash_L A$.

Proof We prove the contrapositive of the claim. Suppose A is a formula such that $\nvdash_{\mathrm{L}} A$. Then, $A \notin \mathrm{L_{TH}}$, and by Lemma 4.12, there is a prime and regular (and a-consistent) L-theory x such that $\mathrm{L_{TH}} \subseteq x$ and $A \notin x$. Then, the canonical L-model is defined and x is a member of O^C in the canonical L-model such that $x \nvDash^C A$. Given that the canonical L-model is an L-model, we have $\nvDash_{\mathrm{L}} A$ by Definition 3.3.

In what follows, we briefly discuss strong soundness and completeness. Consider the following definitions.

Definition 5.5 (Proof-theoretical consequence relations) Let L be a DH-logic, Γ a set of wffs and A a wff. We define: (a) Proof-theoretical consequence relation (first sense): $\Gamma \vdash^1_{\mathrm{L}} A$ iff there is a finite sequence of wffs $B_1, ..., B_m$ such that for each B_i ($1 \leq i \leq m$), one of (i)-(iii) obtains (i) $B_i \in \Gamma$; (ii) B_i is an axiom of L; (iii) B_i is the result of applying one of the rules of inference of L to one or more precedent wffs in the sequence. (b) Proof-theoretical consequence relation (second sense): $\Gamma \vdash^2_{\mathrm{L}} A$ iff there is a finite sequence of wffs $B_1, ..., B_m$ such that for each B_i ($1 \leq i \leq m$), one of (i)-(iv) obtains (i) $B_i \in \Gamma$; (ii) B_i is a theorem of L; (iii) B_i is the result of applying Adj; (iv) B_i is the result of applying L-entailment (L-ent) (L-ent is the following rule: $\vdash_{\mathrm{L}} A \to B \ \& \ A \Rightarrow B$).

Definition 5.6 (Semantical consequence relation) Let L be a DH-logic, Γ a set of wffs and A a wff. We define: $\Gamma \vDash_{\mathrm{L}} A$ iff for any L-model M and $a \in O$, $O \vDash A$ whenever $O \vDash \Gamma$ ($O \vDash \Gamma$ iff $\forall B \in \Gamma\ O \vDash B$).

Let L be a DH-logic. If the sole rules of inference of L are MP and Adj, then standard strong soundness and completeness theorems are provable for

L provided the *Modus Ponens* axiom a4 is an L-theorem. That is, for any set of formulas Γ and formula A we have $\Gamma \vdash^1_{\mathrm{L}} A$ iff $\Gamma \vDash_{\mathrm{L}} A$. However, if a4 is not an L-theorem or L has more primitive rules of inference in addition to Adj and MP (for example, any of the rules Suf, Pref, Con, a11, a43 or a44), but not the corresponding axioms to these rules, then, although standard strong soundness in provable, strong completeness (of sorts) is provable only in the form if $\Gamma \vDash_{\mathrm{L}} A$, then $\Gamma \vdash^2_{\mathrm{L}} A$. The problem with rules of inference in certain weak logics is that it is not possible in general to build prime theories closed under them. Nevertheless, standard strong completeness is in general provable in the said logics if the "disjunctive" version of each rule is added (for example, the disjunctive version of MP is $C \vee A \ \ \&\ \ C \vee (A \to B) \Rightarrow C \vee B$). (Concerning these brief observations on strong soundness and completeness, cf. [14], Chapter 4.)

6 Related systems and possible further work

The logics G3, G3$_{\mathrm{L}}$ and G3$_{\mathrm{DH}}$ are built upon the positive fragment of Gödelian 3-valued logic G3. G3 has an intuitionistic-type negation, G3$_{\mathrm{L}}$ essentially has a De Morgan negation, and G3$_{\mathrm{DH}}$ has a negation of dual intuitionistic type. G3$_{\mathrm{DH}}$ contains all logics investigated in the present paper. It also contains a number of (possibly interesting) logics we have not considered here, which could in principle be studied in a similar way to which the ones described in the paper have been investigated.

Definition 6.1 (The matrix MG3) The matrix MG3 is the structure ($\mathcal{V}, D$, $\mathcal{F}$) where (1) $\mathcal{V} = \{0, 1, 2\}$ and $0 < 1 < 2$; (2) $D = \{2\}$ and (3) $\mathcal{F} = \{f_\to, f_\wedge, f_\vee, f_{\dot\neg}\}$ where $f_\to, f_\vee, f_{\dot\neg}$ are defined according to the following truth-tables:

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

	$\dot\neg$
0	2
1	0
2	0

Definition 6.2 (The matrices MG3$_{\mathbf{L}}$ and MG3$_{\mathbf{DH}}$) The matrices MG3$_{\mathrm{L}}$ and MG3$_{\mathrm{DH}}$ are structures $(\mathcal{V}, D, \mathcal{F})$ where $\mathcal{V}, D$ and $\mathcal{F}$ are defined similarly as in MG3 except for the function for negation, which is defined according to the following truth-tables

MG3$_{\mathrm{L}}$

	$\sim$
0	2
1	1
2	0

MG3$_{\mathrm{DH}}$

	$\neg$
0	2
1	2
2	0

Definition 6.3 (Axiomatization of G3, G3$_{\mathbf{L}}$ and G3$_{\mathbf{DH}}$) Positive intuitionistic logic, H$_+$, can be axiomatized by adding a7 and a27 to B$_+$. Then, G3, G3$_{\mathrm{L}}$ and G3$_{\mathrm{DH}}$ are axiomatized as follows:

1. G3: H_+ plus $(A \to \overset{\bullet}{\neg} B) \to (B \to \overset{\bullet}{\neg} A)$, $\overset{\bullet}{\neg} A \to (A \to B)$ and $(A \vee \overset{\bullet}{\neg} B) \vee (A \to B)$ (cf. [8] and references therein).
2. $\mathrm{G3_L}$: H_+ plus $A \to \sim\sim A$, $\sim\sim A \to A$, $(A \wedge \sim A) \to (B \vee \sim B)$, $(A \vee \sim B) \vee (A \to B)$, $\sim A \to [A \vee (A \to B)]$ and the rule Con, $A \to B \Rightarrow \sim B \to \sim A$ (cf. [9], [11], p.192).
3. $\mathrm{G3_{DH}}$: H_+ plus $(A \to B) \vee (B \to A)$, $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$, $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$, $\neg\neg A \to A$, $A \to (B \vee \neg B)$, $[A \to \neg(C \vee \neg B)] \to [[(B \to A) \to B] \to B]$, $[(A \to B) \wedge \neg(A \to B)] \to (\neg\neg A \wedge \neg B)$ and the rule Con, $A \to B \Rightarrow \neg B \to \neg A$ (cf. [19], [6]; we note that $\mathrm{G3_{DH}}$ is labelled $\mathrm{G3^2}$ in [6] and $\mathrm{G3^{WB}}$ in [19]).

We remark a couple of propositions. Proposition 6.4 displays the relationship the three types of negation maintain to each other. On the other hand, Proposition 6.5 shows that all logics contained in $\mathrm{G3_{DH}}$ are paraconsistent in the sense that there are non-trivial theories containing a formula and its negation. It is interesting to note that the same fact is predicable of $\mathrm{G3_L}$ (not, of course, of G3). In this sense, we note the following fact. Although as seen, $\mathrm{G3_{DH}}$ (and consequently, $\mathrm{CC}\omega$) is paraconsistent, there are weak versions of the ECQ axiom provable even in $\mathrm{CC}\omega_2$ (e.g., $(\neg A \wedge \neg\neg A) \to B$ —cf. Proposition 2.9). In $\mathrm{G3_L}$, however, the ECQ axiom is generally invalid. (Note that $\mathrm{G3_L}$ drops PEM but has both double negation axioms.)

Proposition 6.4 (Relationship between $\overset{\bullet}{\neg}$, $\sim$, $\neg$) *Consider the matrix MG3 with functions for negation $f_\sim$ and $f_\neg$ added. Then, we have that for any wff A, $\overset{\bullet}{\neg} A \to \sim A$ and $\sim A \to \neg A$ are valid, but the converses are not.*

Proof It is immediate.

Proposition 6.5 (non-trivial inconsistent theories) *Let L be an DH-logic. There are prime, regular and inconsistent L-theories (i.e., theories containing a wff and its negation) that are not trivial.*

Proof Consider the set $z = \{B \mid \vdash_{\mathrm{L}} A \ \ \& \ \ \vdash_{\mathrm{L}} [A \wedge (p_i \wedge \neg p_i)] \to B\}$. It is easy to show that z is a regular L theory and that it is inconsistent. Anyway, z is not trivial. Let p_i, p_m $(i \neq m)$ be propositional variables. Consider any assignment v defined in $\mathrm{MG3_{DH}}$ such that $v(p_i) = 1$ and $v(p_m) = 0$. Clearly, $v[A \wedge (p_i \wedge \neg p_i)] = 1$, but $v[[A \wedge (p_i \wedge \neg p_i)] \to p_m] = 0$, whence by the soundness theorem of $\mathrm{G3_{DH}}$, we get $\nvdash_{\mathrm{G3_{DH}}} [A \wedge (p_i \wedge \neg p_i)] \to p_m$. Consequently, $p_m \notin z$. Then, we apply Lemma 4.12, and there is a prime, regular and a-consistent theory x such that $z \subseteq x$ and $p_m \notin x$. Therefore x is inconsistent but not trivial.

A Appendix

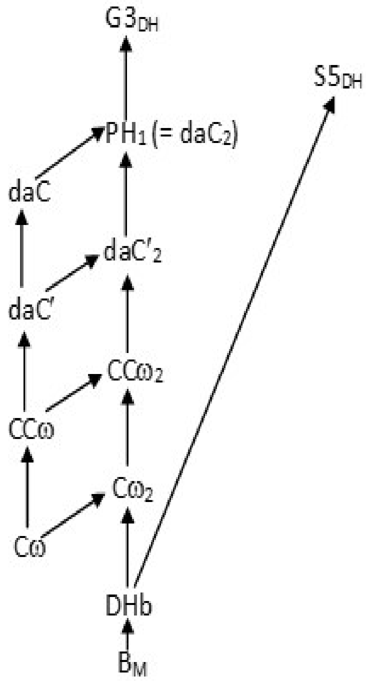
The logic $\mathrm{C}\omega$ is axiomatized as follows (cf. [3], [16], Definition 2.4 and Lemma 2.13):

Axioms: a7, a27, a30, A2, A4 and A5, PEM and DNE.

Rule of inference: MP

Then, we define the following extensions of Cω (cf. [6]): CCω: Cω plus Con; daC$'$: Cω plus a43; daC: CCω plus a44; PH$_1$: daC plus a42.

Then, the logics Cω_2, CCω_2, daC$'_2$ and daC$_2$ are the result of adding A7, $(\neg A \wedge \neg B) \to \neg(A \vee B)$, to C$\omega$, CC$\omega$, daC$'$ and daC, respectively. The relations these logics maintain to each other are summarized in the following diagram (for any logics L, L$'$, L $\to$ L$'$ means that L$'$ is an extension of L, but not conversely).



The logic H$_+$ is axiomatized with a7, a27, a30, A2, A4, A5 and MP. The logics B$_{\rm M}$ and G3$_{\rm DH}$ are defined in section 2 and 6, respectively.

Proposition A.1 (A7 is not CCω valid) *The De Morgan law A7,* $(\neg A \wedge \neg B) \to \neg(A \vee B)$*, is not CC$\omega$-valid.*

Proof We use the semantics defined by Sylvan in his paper [16]. Let p_i, p_m be distinct propositional variables and M a CCω-model where $a, b, c \in K, Sab$ and Sac and v an assignment such that $v(p_i, a) = v(p_m, a) = 1$; $v(p_i, b) = 0, v(p_m, b) = 1$; $v(p_i, c) = 1, v(p_m, c) = 0$. Then, $v(\neg p_i, a) = 1, v(\neg p_m, a) = 1$, whence $v(\neg p_i \wedge \neg p_m, a) = 1$. But $v(\neg(p_i \vee p_m), a) = 0$ since $v(p_i \vee p_m, a) = 1, v(p_i \vee p_m, b) = 1$ and $v(p_i \vee p_m, c) = 1$. Consequently, $v[(\neg p_i \wedge \neg p_m) \to \neg(p_i \vee \neg p_m), a] = 0$.

Finally, the matrix MS5$_{\rm DH}$ determining the logic S5$_{\rm DH}$ is defined. The logic S5$_{\rm DH}$ is a 3-valued extension of positive modal logic S5$_+$ (cf. [5]). S5$_{\rm DH}$ is not included in G3$_{\rm DH}$ but it could be interpreted, as well as its subsystems, with a reduced RM-semantics similarly as G3$_{\rm DH}$ and the subsystems of this logics considered in the present paper have been interpreted.

Definition A.2 (The matrix $\mathbf{MS5_{DH}}$) The matrix $\mathrm{MS5_{DH}}$ is the structure $(\mathcal{V}, D, \mathcal{F})$ where $\mathcal{V}, D, \mathcal{F}$ are defined similarly as in $\mathrm{MG3_L}$, except that now $D = \{1, 2\}$ and $f_{\to}$ is defined according to the following truth table:

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

Concerning the logic $\mathrm{S5_{DH}}$, that is, the logic determined by $\mathrm{MS5_{DH}}$, we have not axiomatized it and we ignore if it has been axiomatized somewhere in the literature. However, we remark that two logics related to it, i.e., the logic determined by $\mathrm{MS5_{DH}}$ when 2 is the only designated value, and the logic determined by the matrix resulting of replacing the truth-table $\overset{\bullet}{\neg}$ for negation with truth-table $\sim$, have been axiomatized in [19] and [13], respectively.

We remark that the following items in Definition 2.13 are verified by $\mathrm{MS5_{DH}}$: a1-a13, a20-a26, a32, a33, a35-a39, a41-a43. But, of course, there are many other theses and rules verified by $\mathrm{MS5_{DH}}$, which are not provable in $\mathrm{G3_{DH}}$, for instance, *disjunctive Peirce's law*, i.e., $A \vee (A \to B)$. It is worth-remarking that $\mathrm{S5_{DH}}$ and all the logics included in it can be proved paraconsistent similarly as DH-logics are shown in Proposition 6.5.

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