pa33. Raa^*a^*

pa34. Ra^*aa^*

pa35. Raa^*a^{**}

pa36. $Rabc \Rightarrow Rac^*b^*$

pa37. $Rabc \Rightarrow a \leq b^*$

pa38. $Rabc \Rightarrow b \leq a^*$

pa39. $Ra^*a^*a^*$

pa40. $(Rabc \ \& \ a \in O) \Rightarrow (b \leq a \text{ or } a \leq c^*)$

pa41. $Rabc \Rightarrow (b \leq a^* \text{ or } a \leq c)$

pa42. $Ra^*bc \Rightarrow (b \leq a \text{ or } a^* \leq c)$

pa43. $Ra^*bc \Rightarrow (b \leq a^* \text{ or } a \leq c)$

pa44. $Rabc \Rightarrow c^* \leq a^*$

pa45. $(Rabc \ \& \ Ra^*de) \Rightarrow (d \leq c \text{ or } b \leq e)$

pa46. $Ra^*bc \Rightarrow a^* \leq c$

pa47. $Rabc \Rightarrow (a \leq b^* \text{ or } a \leq c)$

pa48. $Rabc \Rightarrow (a \leq b^* \text{ or } c^* \leq a^*)$

pa49. Ra^*aa

pa50. $a \in O \Rightarrow a^* \leq a$

Lemma 3.11 (EHb-validity of a1-a50)**.** *Let $\mathcal{M}$ be a class of EHb-models and $M \in \mathcal{M}$. Then, for any k ($1 \leq k \leq 10$; or $12 \leq k \leq 50$) ak is true in M if pak holds in M; and a11 preserves truth in M if pa11 holds in M.*

Proof. The proof of the validity of a1-a32, a32′ (that a11 preserves truth in M, in the case of a11) can be found in Robles & Méndez (2018). The proof of a33-50 is similar to that given in Routley et al. (1982, Chapter 4), for extensions of Routley and Meyer's basic logic B. Let us prove some cases:

(a) *a34, $(A \wedge \neg B) \to \neg(A \to B)$, is true in M*: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \vDash A \wedge \neg B$ but (2) $a \nvDash \neg(A \to B)$. By 1, we have (3) $a \vDash A$ and (4) $a^* \nvDash B$; and by 2, (5) $a^* \vDash A \to B$. Then, we apply pa34 (6) Ra^*aa^*, and we conclude (7) $a^* \vDash B$ by 3, 5 and 6, contradicting 4.

(b) *a38, $\neg A \to (A \to B)$, is true in M*: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \vDash \neg A$ but (2) $a \nvDash A \to B$. By 1, we get (3) $a^* \nvDash A$, and by 2, there are $b, c \in K$ in M such that (4) $Rabc$, (5) $b \vDash A$ and (6) $c \nvDash B$. By pa38 and 4, we get (7) $b \leq a^*$, whence by 5, we obtain (8) $a^* \vDash A$, contradicting 3.

(c) *a40, $(A \vee \neg B) \vee (A \to B)$, is true in M*: for reductio suppose that there are wffs A, B and $a \in O$ such that (1) $a \nvDash A$, (2) $a \nvDash \neg B$ (i.e, $a^* \vDash B$) and (3) $a \nvDash A \to B$. By 3, there are $b, c \in K$ such that (4) $Rabc$, (5) $b \vDash A$ and (6) $c \nvDash B$. By pa40 and 4 (given that $a \in O$), we have (7) $b \leq a$ or (8) $a^* \leq c$. But 5 and 7 contradict 1, whereas 2 and 8 contradict 6.

(d) *a45, $\neg(A \to B) \to (B \to A)$, is true in M*: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \vDash \neg(A \to B)$, i.e., (2) $a^* \nvDash A \to B$, but (3) $a \nvDash B \to A$. By 2, there are $b, c \in K$ in M such that (4) Ra^*bc, (5) $b \vDash A$ and (6) $c \nvDash B$. By 3, there are $d, e \in K$ in M such that (7) $Rade$, (8) $d \vDash B$ and (9) $e \nvDash A$. But by pa45, we have either (10) $d \leq c$ or (11) $b \leq e$. Then, 8 and 10 contradict 6; and 5 and 11 contradict 9. □

Definition 3.12 (L-models). Let L be an EHb-logic. An L-model is defined when adding to Hb-models the semantical postulates corresponding to the items in Definition 2.12 added to Hb for axiomatizing L. For example, consider the extension of Hb axiomatized by a7 and a26 Then, an EHb-model for this system is a structure $(O, K, R, *, \vDash)$ where O, K, R, $*$, and $\vDash$ are defined exactly as in Definition 3.1, save for the addition of the postulates pa7 and pa26. (The notion of L-validity is defined according to the general Definition 3.3. Notice that the system just defined is the expansion of the positive fragment of Lewis' logic S4 (cf. Hacking, 1963) with the basic quasi-Boolean H-negation defined above).

Theorem 3.13 (Soundness of EHb-logics). *Let L be an EHb-logic. For any wff A, if* $\vdash_L A$*, then* $\vDash_L A$*.*

Proof. By Corollary 3.8 and Lemma 3.11, given Definition 3.12. □

The section is ended with the following remark (cf. Remark 2.13).

Remark 3.14 (Bb-models). We note that Bb-models, RM-models for B-negation, are defined by adding the postulate $a^* \leq a$ to Hb-models. Bb-models characterize Hb plus the axioms $C \to [B \to (A \vee \neg A)]$ (A9′) and $C \to [\neg(A \vee \neg A) \to B]$ (A10′).

4. Completeness. Preliminary notions and lemmas

We use a canonical model construction in order to prove the completeness of the EHb-logics. The canonical model is defined and then we prove that if A is not a theorem of the logic in question, then A is false in at least a designated point in the canonical model. Firstly, we define some preliminary notions needed to define the canonical model.

Definition 4.1 (EHb-theories). Let L be an EHb-logic. An L-theory is a set of wffs closed under Adjunction (Adj) and L-entailment (L-ent). That is, a is an L-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of L and $A \in a$, then $B \in a$.

By the term EHb-theory, we will generally refer to any theory defined upon an EHb-logic as just indicated. The classes of EHb-theories of interest in the present paper are remarked in the following definition.

Definition 4.2 (Classes of EHb-theories). Let L be an EHb-logic and a an L-theory. We set: (1) a is *prime* iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$. (2) a is *empty* iff it contains no wffs. (3) a is *regular* iff a contains all theorems of L. (4) a is *trivial* iff every wff belongs to it. (5) a is *a-consistent* (consistent in an absolute sense) iff a is not trivial.

Next, we prove two useful propositions.

Proposition 4.3 (On non-empty EHb-theories). *Let L be an EHb-logic and a a non-empty L-theory. Then,* $\neg(A \wedge \neg A) \in a$ *for any wff A.*

Proof. Immediate by A9. □

Proposition 4.4 (a is a-inconsistent iff $A \wedge \neg A \in a$). *Let L be an EHb-logic and a* *an L-theory. Then, a is a-inconsistent iff $A \wedge \neg A \in a$ for some wff A.*

Proof. Immediate by T3 (ECQ). □

Notice then that, within the context of Hb and its extensions, a-consistency and (negation) consistency in the customary sense of the term coincide.

Next, canonical models are defined.

Definition 4.5 (Canonical EHb-models)**.** Let L be an EHb-logic and K^T be the set of all L-theories. Then, the ternary relation R^T is defined in K^T as follows: for any $a, b, c \in K^T$, $R^T abc$ iff for any wffs A, B, $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Next, let K^C be the set of all prime, non-empty and a-consistent L-theories, O^C be the subset of K^C formed by all regular L-theories and R^C be the restriction of R^T to K^C. On the other hand, let $*^C$be defined on K^C as follows: for all $a \in K^C$, $a^{*^C} = \{A \mid \neg A \notin a\}$. Finally, the relation $\vDash^C$is defined as follows: for each formula A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the structure $(O^C, K^C, R^C, *^C, \vDash^C)$ is the canonical L-model.

As pointed out above, completeness leans upon proving that the canonical L-model is indeed an L-model. Now, the proof of this fact is similar to the corresponding proofs in standard relevant logics (cf. Routley et al., 1982, Chapter 4), except for an important difference: everytime an L-theory a is built, a has to be proven non-empty and a-consistent. But this is easily shown by using Propositions 4.3 and 4.4.

In the sequel, a series of preliminary lemmas follows. We suppose we are given an EHb-logic L: some of the lemmas below are not provable for weaker logics.

Lemma 4.6 (On the relation R^T)**.** *Let a, b be non-empty L-theories and c be an a-consistent and prime L-theory. Then,*

(a) The set $x = \{B \mid \exists A(A \to B \in a \ \ \& \ \ A \in b)\}$ is a non-empty L-theory such that $R^T abx$.

(b) If $R^T abc$, then there is a prime and a-consistent (and non-empty) prime L-theory x such that $a \subseteq x$ and $R^T xbc$.

(c) If $R^T abc$, then there is a prime and a-consistent (and non-empty) prime L-theory x such that $b \subseteq x$ and $R^T axc$.

Proof. (a) It is easy to prove that x is an L-theory such that $R^T abx$. Then, the non-emptiness of x follows by A9 (notice that the simpler thesis $B \to \neg(A \wedge \neg A)$ is not sufficient).

(b) A prime L-theory a such that $a \subseteq x$ and $R^T xbc$ is built up similarly as in standard relevant logics. Next, x is immediately shown a-consistent. If x is a-inconsistent, then $A \to B \in x$, for $A \in b$ and arbitrary wff B, whence c would be a-inconsistent, contradicting one of the hypotheses.

(c) As in the precedent case, a prime L-theory x such that $b \subseteq x$ and $R^T axc$ is built up similarly as in relevant logics. Then, the a-consistency of x is proven as follows. Suppose x is a-inconsistent. Let $A \in a$ and B be an arbitrary wff. Then, $C \wedge \neg C \in x$ for some wff C, by Proposition 4.4. By A10, $A \to [(C \wedge \neg C) \to B]$ is an L-theorem. So, $(C \wedge \neg C) \to B \in a$, whence $B \in c$, by $R^T axc$, contradicting the a-consistency of c (notice that the ECQ axiom is not sufficient). □

Lemma 4.7 ($*^C$ is an operation on K^C)**.** *Let a be a prime, non-empty and a-consistent L-theory. Then, a^{*^C} is a prime, non-empty and a-consistent L-theory as well.*

Proof. As there is no danger of confusion between a^* in K and the canonical L-theory a^{*^C} in K^C, we omit the supersript ^{C} above $*$ in this proof and the proofs to follow. Now, a^* is proven a prime L-theory similarly as in standard relevant logics. Then a^* is shown non-empty and a-consistent as follows. (1) a^* is non-empty. Let $A \in a$. If a^* is empty, then $A \notin a^*$, whence $A \wedge \neg A \in a$, contradicting the a-consistency of a. (2) a^* is a-consistent. If a^* is a-inconsistent, then $A \wedge \neg A \in a^*$ for some wff A, whence $\neg(A \wedge \neg A) \notin a$, contradicting the non-emptiness of a (cf. Proposition 4.3). □

Lemma 4.8 ($a \leq^C b$ iff $a \subseteq b$)**.** *For any $a, b \in K^C$, $a \leq^C b$ iff $a \subseteq b$. (By d1, $a \leq^C b$ is equivalent to $R^C xab$, for some $x \in O^C$.)*

Proof. It is proved similarly as in standard relevant logics, except that in addition to being regular, x has to be proven a-consistent, which is immediate by Lemma 4.6(b). □

Finally, we prove the primeness lemma and then the adequacy of the canonical valuation relation.

Lemma 4.9 (Extension to prime L-theories)**.** *Let a be an L-theory and A a wff such that $A \notin a$. Then, there is a prime L-theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. By direct application of Kuratowski-Zorn's Lemma as in Routley et al. (1982, Chapter 4, pp. 310-311). □

Lemma 4.10 ($\vDash^C$ and clauses (i)-(v))**.** *For any $a, b, c \in K^C$ and wffs A, B: (i) $(a \leq^C b \ \&\ a \vDash^C p) \Rightarrow b \vDash^C p$; (ii) $a \vDash^C A \wedge B$ iff $a \vDash^C A$ and $a \vDash^C B$; (iii) $a \vDash^C A \vee B$ iff $a \vDash^C A$ or $a \vDash^C B$; (iv) $a \vDash^C A \to B$ iff for all $b, c \in K^C$, $(R^C abc \ \&\ b \vDash^C A) \Rightarrow c \vDash^C B$; (v) $a \vDash^C \neg A$ iff $a^{*^C} \nvDash^C A$.*

Proof. Similar to those in relevant logics, save that Lemma 4.6 (cases a and c) is used to prove non-emptiness and a-consistency when required. □

Notice that Lemma 4.6 (cases a and c) and Lemma 4.10 require A9 and A10 in order to be proven.

5. Completeness of the EHb-logics

By using the primeness lemma (Lemma 4.9), it is easy to prove that if A is not an L-theorem, then there is some prime, regular and a-consistent L-theory not containing A. Then, for proving the completeness of L, it only remains to show that the canonical L-model is in fact a model. Now, given Lemma 4.7 ($*^C$ is an operation on K^C) and Lemma 4.10 (Adequacy of the canonical evaluation clauses), it only remains to prove (1) the set O^C is non-empty; and (2) the postulates hold canonically.

Corollary 5.1 (O^C is not empty)**.** *Let $(O^C, K^C, R^C, *^C, \vDash^C)$ be the canonical L-model. Then, the set O^C is not empty.*

Proof. Clearly, L is a-consistent since all axioms and rules of L are theorems and rules of classical propositional logic when read with the classical connectives. Then, Corollary 5.1 is immediate by Lemma 4.9. □

Lemma 5.2 (The postulates are canonically valid)**.** *Let L be an EHb-logic. Then, (1) P1, P2a, P2b, P2c, P2d, P3 and P4 hold in all canonical EHb-models. (2) pak holds in the canonical L-model if ak is provable in L (*$1 \leq k \leq 50$*).*

Proof. The proof is similar to that provided in Routley et al. (1982, Chapter 4), for extensions of Routley and Meyer's basic logic B. Actually, a proof for P1, P2a, P2b, P2c, P2d and P3 can be found in the aforementioned chapter and P4 is proved below. Then, pa$k$ holds in the canonical L-model if ak is provable in L ($1 \leq k \leq 50$). Now, concerning pa1-pa32 and pa32$'$, the proof can be found in Robles & Méndez (2018). And concerning pa33-pa50, we prove the canonical validity of the postulates used in Lemma 3.11, as a way of an example.

(a) *P4, $a \leq a^*$, is provable in the canonical L-model*: Suppose $a \in K^C$ and (1) $A \in a$. We have to prove $A \in a^*$. Suppose (2) $A \notin a^*$. Then, (3) $\neg A \in a$ whence by 1, we get (4) $A \wedge \neg A$, contradicting the a-consistency of a (cf. Proposition 4.4).

(b) *pa34, Ra^*aa^*, is provable in the canonical L-model:* Let $a \in K^C$ and suppose (1) $A \to B \in a^*$ and (2) $A \in a$. We have to prove $B \in a^*$. By 1, we get (3) $\neg(A \to B) \notin a$, whence, by a34, $(A \wedge \neg B) \to \neg(A \to B)$, we have (4) $A \wedge \neg B \notin a$, i.e., (5) $A \notin a$ or $\neg B \notin a$. Then, (6) $\neg B \notin a$ follows by 2 and, finally, (7) $B \in a^*$ by 6, as was to be proved.

(c) *pa38, $Rabc \Rightarrow b \leq a^*$ is provable in the canonical L-model*: Let $a, b, c \in K^C$ and suppose (1) $R^C abc$ and (2) $A \in b$. We have to prove $A \in a^*$. Let B an arbitrary wff and for reductio suppose (3) $A \notin a^*$, i.e., $\neg A \in a$. By a38 we have (4) $\neg A \to (A \to B)$, whence by 3, we get (5) $A \to B \in a$ and, finally, by 1, 2 and 5, (6) $B \in c$, contradicting the a-consistency of c.

(d) *pa40, $(Rabc \ \ \& \ \ a \in O) \Rightarrow (b \leq a$ or $a^* \leq c)$, is provable in the canonical L-model:* Let $a \in O^C$, $b, c \in K^C$ and suppose (1) $R^C abc$ and, for reductio, (2) $A \in b$, (3) $A \notin a$, (4) $B \in a^*$, i.e., $\neg B \notin a$ and (5) $B \notin c$. By a40, we have (6) $(A \vee \neg B) \vee (A \to B) \in a$, given that $a \in O^C$. Then, by 3, 4 and 6, we get (7) $A \to B \in a$, whence by 1 and 2, (8) $B \in c$ is obtained, contradicting 5.

(e) *pa45, $(Ra^*bc \ \ \& \ \ Rade) \Rightarrow (d \leq c$ or $b \leq e)$, is provable in the canonical L-model:* Let $a, b, c \in K^C$ and suppose (1) $R^C a^*bc$, (2) $Rade$ and, for reductio, (3) $A \in d$, (4) $A \notin c$, (5) $B \in b$ and (6) $B \notin e$. By 1, 4 and 5, we have (7) $B \to A \notin a^*$, i.e., (8) $\neg(B \to A) \in a$. By a45, (9) $\neg(B \to A) \to (A \to B)$ is an L-theorem. Then, we get (10) $A \to B \in a$, by 8 and 9. Finally, by 2, 3 and 10, (11) $B \in e$ is derivable, contradicting 6. □

Proposition 5.3 (The canonical model is a model)**.** *Let L be an EHb-logic. The canonical L-model is indeed an L-model.*

Proof. Given Definition 4.5 and Corollary 5.1, the proof follows by Lemma 4.7 ($*^C$is a operation on K^C), Lemma 4.10 (Adequacy of the canonical clauses) and Lemma 5.2 (The postulates hold canonically). □

Theorem 5.4 (Completeness of the EHb-logics)**.** *Let L be an EHb-logic. For any wff A, if* $\vDash_L A$*, then* $\vdash_L A$*.*

Proof. We prove the contrapositive of the claim. Suppose A is a formula such that $\nvdash_{\mathrm{L}} A$ and let L be the set of all its theorems. Then, $A \notin \mathrm{L}$, and by Lemma 4.9, there is a prime and regular (and a-consistent) L-theory x such that $\mathrm{L} \subseteq x$ and $A \notin x$. Then, the canonical model is defined and x is a member of O^C in the canonical L-model such that $x \nvDash^C A$. Given that the canonical L-model is an L-model, we have $\nvDash_{\mathrm{L}} A$ by

Definition 3.3. □

6. A note on reduced Routley-Meyer semantics and strong completeness

As pointed out in the introduction of this paper, there are two types of RM-semantics, RM_0-semantics and RM_1-semantics. The latter can be reduced or unreduced RM_1-semantics. In Brady (2003) and Routley et al. (1982), it is explained at length why reduced RM_1-semantics is preferable when it is possible to define it.

Well then, a necessary condition for defining a reduced RM_1-semantics for a given logic L is that it is possible to build up prime regular theories closed under the primitive rules of inference of L. But this necessary condition is not generally met by weak logics. Let us elaborate on the question. Consider the logics C, C′ defined below (cf. Routley et al., 1982, p. 289 —the logic C′ is defined by us).

Definition 6.1 (The logic C)**.** The logic C is axiomatized by adding to B (cf. Proposition 2.8) the following axioms: a3 (Axiom Suffixing), a4 (Axiom Modus Ponens) and a36 (Axiom Contraposition) (cf. Definition 2.12; we note that a2 (Axiom Prefixing) is derivable in C by a3, a36 and the Double Negation Axioms).

Definition 6.2 (The logic C′)**.** The logic C′, a sublogic of C, is axiomatized by adding to B_M the axioms a2, a3, a4 and a36.

In Routley et al. (1982, Chapter 4), it is essentially shown that C and any of its extensions with Adj and MP as the sole primitive rules of inference can be given a reduced RM_1-semantics. Although we cannot prove it here, it could be shown that this result still holds for the weaker logic C′ and its extensions with only Adj and MP as primitive rules of inference. Consequently, all extensions of Hb included in G3 and/or $S5_{G3}$ with a2, a3, a4, a36 and Adj and MP as the only primitive rules of inference could be endowed with a reduced RM_1-semantics instead of the unreduced one provided for them in the present paper. Moreover, even logics weaker than C′ can be given a reduced RM_1-semantics on the condition that each one of their primitive rules of inference (except Adj) be accompanied by its disjunctive version or the corresponding "thesis form" (cf. Routley et al., 1982, pp. 356, ff.; for example, the disjunctive version of the rule Suf is $D \vee (A \to B) \Rightarrow D \vee [(B \to C) \to (A \to C)]$, while the corresponding thesis form is, of course, a3). Consequently, any extension L of Hb included in G3 and/or $S5_{G3}$ with any (or all) the rules MP, Suf, Pref, Con and a11 as primitive rules of inference can be given a reduced RM_1-semantics with the proviso that the disjunctive version or the thesis form of the rules present be added.

Related in a way, that we cannot pause here to explain with some detail, to the possibility of defining a reduced RM_1-semantics for a given logic L is that of proving a standard strong completeness theorem for L. In general, if L is a logic with other primitive rules of inference than Adj and MP and lacking the disjunctive versions or the thesis forms of the primitive rules in question, then strong completeness (of sorts) can only be proven w.r.t. a deductive consequence relation $\vdash_L$ defined as follows: for any set of wffs Γ and wff A, $\Gamma \vdash_L A$ iff there is a finite sequence of wffs $B_1, ...B_m$ such that for each B_i $(1 \leq i \leq m)$, we have (1) $B_i \in \Gamma$; (2) B_i is a theorem of L; (3) B_i is the result of applying the rule Adj; (4) B_i is the result of applying the rule L-entailment (L-ent) to two precedent wffs in the sequence (L-ent is the following rule: $\vdash_L A \to B \ \ \& \ \ A \Rightarrow B$). Nevertheless, a standard strong completeness theorem is provable for L provided the disjunctive versions or the thesis forms of the primitive rules of inference of L are added to its axiomatization.

7. Concluding remarks

It was pointed out in Proposition 2.9 and in §4 that axiomatization of H-negation (as this notion has been understood in the present paper) requires A9 and A10 in general, since the ECQ axiom, $(A \wedge \neg A) \rightarrow B$, is insufficient in the case of certain logics. Nevertheless, the ECQ axiom could be sufficient in logics built on expansions of the language upon which Hb and its extensions have been defined here. In particular, if fusion ($\circ$) and the 'left implication' ($\leftarrow$) of the Lambeck calculus (cf. e.g., Restall, 2000a) are added as new connectives, then A9 and A10 can be derived from $\mathrm{B_M}$ plus the ECQ axiom (we owe this remark to an anonymous referee of the JANCL). Maybe this fact would also obtain in the case of addition of other connectives, such as those investigated in Bimbó & Dunn (2008). Thus, future work on the topic could consist in the restructuration of the systems investigated above now based on a new definition of Hb built upon the expanded language with $\circ$ and $\leftarrow$, and/or with other connectives as well.

In the present paper, we have defined a family of logics including Sylvan and Plumwood's logic $\mathrm{B_M}$ extended with an intuitionistic negation of sorts, essentially axiomatized by A9 and A10. Of course, there is an immense literature on the two topics central to the investigation here reported, intuitionistic logics and the notion of negation (cf., e.g., Moschovakis, 2018 and Horn & Wansing, 2020, and references in these two items), but we think that the perspective we have adopted is still new. Let us briefly comment on work related to the one recorded in the preceding pages. Generally speaking, the literature can be divided in two types of investigations: (a) negation expansions of systems built upon a minimal consequence relation and without using the conditional connective. For instance, one of the aforesaid minimal systems can be found in Dunn (2000). It has the rule $A \wedge \neg A \vdash B$, or alternatively, $B \vdash A \vee \neg A$. Or to take another example, in Shramko (2005), the rules assumed are $A \wedge \neg A \vdash B$ and $A \vdash \neg\neg A$, or alternatively, $B \vdash A \vee \neg A$ and $\neg\neg A \vdash A$ (cf. also Restall, 2000b; Restall, 1999; and Dunn, 1993); (b) negation expansions built upon positive systems with a strong conditional. For instance, the positive fragments of relevant logic, intuitionistic logic and both Gödelian 3-valued logic G3 and Lewis' S5 are the starting point in Dunn (2000), Wansing (2016) and Yang (2012), respectively. In this sense, we think that the point of view we have adopted in the present paper is not only new within the semantics used (RM-semantics), but also in what concerns the family of logics studied with the said semantics.

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Appendix A. Appendix

Proposition A.1 (Ant, DNE are deriv. in FDE_+ plus ECQ & CPEM)**.** *The axiom DNE, $\neg\neg A \to A$, and the rule Ant, $(A \wedge B) \to \neg C \Rightarrow (A \wedge C) \to \neg B$ are derivable in FDE_+ plus the axioms ECQ, $(A \wedge \neg A) \to B$, and CPEM, $B \to (A \vee \neg A)$.*

Proof. (Sketch)

(a) $(A \wedge B) \to \neg C \Rightarrow (A \wedge C) \to \neg B$:

Suppose (1) $(A \wedge B) \to \neg C$ (Hyp) and (2) $(C \wedge \neg C) \to \neg B$ (ECQ). By 1, 2 and FDE_+, we have (3) $[(A \wedge C) \wedge B] \to \neg B$. On the other hand, we obviously have (4) $[(A \wedge C) \wedge \neg B] \to \neg B$. By 3, 4 and FDE_+, we get (5) $[(A \wedge C) \wedge (B \vee \neg B)] \to \neg B$. Now, we use (6) $C \to (B \vee \neg B)$ (CPEM), whence we obtain (7) $[(A \wedge C) \wedge (A \wedge C)] \to [(A \wedge C) \wedge (B \vee \neg B)]$. Finally, by 5, 7 and FDE_+, we get (8) $(A \wedge C) \to \neg B$, as was to be proved.

(b) $\neg\neg A \to A$:

We have (1) $\neg\neg A \to (A \vee \neg A)$ (CPEM). By 1 and FDE_+, we have (2) $(\neg\neg A \wedge \neg\neg A) \to [(\neg\neg A \wedge A) \vee (\neg\neg A \wedge \neg A]$. We use now (3) $(\neg\neg A \wedge \neg A) \to A$ (ECQ). By 3 and FDE_+, we get (4) $[(\neg\neg A \wedge A) \vee (\neg\neg A \wedge \neg A)] \to A$. Finally, by 2, 4 and FDE_+, we have (5) $\neg\neg A \to A$, as was to be proved. □

Consider the following definitions.

Definition A.2 (The matrix MG3)**.** The matrix MG3 is the structure $(\mathcal{V}, D, \mathcal{F})$ where (1) $\mathcal{V} = \{0, 1, 2\}$ and $0 < 1 < 2$; (2) $D = \{2\}$ and $\mathcal{F} = \{f_\to, f_\wedge, f_\vee, f_\neg\}$ where $f_\to, f_\vee, f_\neg$ are defined according to the following truth-tables:

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

	$\neg$
0	2
1	0
2	0

Definition A.3 (The matrix $MS5_{G3}$)**.** The matrix $MS5_{G3}$ is the structure $(\mathcal{V}, D, \mathcal{F})$ where $\mathcal{V}$, D and $\mathcal{F}$ are defined similarly as in MG3, except that now $D = \{1, 2\}$ and $f_\to$ is defined according to the following truth-table:

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

Notice that CPEM and DNI are falsified by both MG3 and $MS5_{G3}$.

Definition A.4 (Axiomatization of G3 and $S5_{G3}$)**.** Positive intuitionistic logic, H_+, can be axiomatized by adding a7 and a27 to B_+ (cf. Definitions 2.4 and 2.12); on the other hand, positive modal logic S5 can be axiomatized by adding a7, a26 and a32′ to B_+ (cf. Hacking, 1963). Then, G3 is axiomatized as follows: H_+ plus DNI, a36, a38 and a40 (cf. Robles, 2014, and references therein). Concerning the logic $S5_{G3}$, that is, the logic determined by $MS5_{G3}$, we have not axiomatized it and we ignore if it has been axiomatized somewhere in the literature. However, we remark that two logics related to it, the logics determined by the matrices resulting of changing the truth-table for negation in $MS5_{G3}$ for the ones below have been axiomatized in Robles & Méndez (2019) and Yang (2012), respectively.

	$\neg$
0	2
1	1
2	0

	$\neg$
0	2
1	2
2	0