A basic quasi-Boolean logic of intuitionistic character

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ABSTRACT

The logic $B_{\rm M}$ is Sylvan and Plumwood's minimal De Morgan logic. The aim of this paper is to investigate extensions of $B_{\rm M}$ endowed with a quasi-Boolean negation of intuitionistic character included in 3-valued logic G3 and/or 3-valued logic S5_G3. Unreduced Routley-Meyer ternary relational semantics are provided for all the logics defined in the paper.

KEYWORDS

De Morgan logics; quasi-Boolean logics; paraintuitionistic logics; superintuitionistic logics; Routley-Meyer ternary relational semantics.

1. Introduction

This paper is a preliminary study of quasi-Boolean negation (QB-negation) of intuitionistic character in the context of the Routley-Meyer ternary relational semantics (RM-semantics). As it is known, "possible worlds" ("set-ups" or whatever the name is preferred) can be inconsistent, incomplete or both in standard RM-semantics, that is, in RM-semantics for relevant logics (cf. Routley, Meyer, Plumwood, & Brady, 1982; Brady, 2003, and references therein). However, in this paper, we focus on RM-semantics with models whose elements are always consistent but not necessarily complete possible worlds, as it is the case with (binary relational) Kripke models for propositional intuitionistic logic.

In RM-semantics, negation is customarily interpreted with the Routley operator or Routley star (*), which is adequate for modelling only De Morgan negation (DMnegation) and extensions thereof in the sense that it cannot fail to validate, by virtue of its own definition, the De Morgan laws (i.e., $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$ and $\neg(A \land B) \leftrightarrow (\neg A \lor \neg B)$ and the contraposition rule (i.e., $A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$ —cf. Definitions 2.1-2.3 on the logical language used in the paper and related preliminary notions). Consequently, the type of QB-negation we are going to investigate can be considered from an intuitive point of view, as a DM-negation of an intuitionistic, or better, "paraintuitionistic" or "superintuitionistic" character. (An anonymous referee of the JANCL points out that "paracomplete" could be better suited than "intuitionistic" to characterize the family of logics considered.) Let us look into it with more detail. We begin by examining how "positive" (i.e., negationless) relevant logic can

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proof-theoretically be expanded with Boolean negation (B-negation).

B-negation can be introduced in a positive relevant logic L₊ by adding to it the axiom "Double Negation Elimination" (DNE), $\neg \neg A \rightarrow A$, together with the rule "Antilogism" (Ant) $(A \wedge B) \rightarrow \neg C \Rightarrow (A \wedge C) \rightarrow \neg B$. (Cf. Routley et al., 1982, p. 371; Meyer & Routley, 1973; Meyer & Routley, 1974.)

In Routley et al. (1982, pp. 371-372), it is proved that the "E Contradictione Quodlibet" axiom (ECQ), $(A \land \neg A) \to B$, and the "Conditioned Principle of Excluded Middle" (CPEM), $B \to (A \lor \neg A)$, are theorems of any relevant logic L including Routley and Meyer's basic positive logic B_+ (cf. Definition 2.4, below) plus DNE and Ant. But it would not be difficult to show that the proof provided by Routley et al. could be carried out within the weaker (than B_+) positive fragment, FDE₊, of Anderson and Belnap's *First Degree Entailment logic*, FDE (cf. Anderson & Belnap, 1975, pp. 158, ff.). Moreover, in Proposition A1 in the appendix, it is proven that DNE and Ant are derivable within FDE₊ plus the axioms ECQ and CPEM. Consequently, positive relevant logics can equivalently be expanded with B-negation by adding to FDE₊ either DNE and Ant or else ECQ and CPEM.

Thus, we see, the axioms ECQ and CPEM are the two pillars upon which B-negation can be built given such a weak positive logic as FDE_+ . From the point of view of possible-worlds semantics, the ECQ axiom can be viewed as an expression of the thesis that all possible-worlds are consistent (no possible-world contains a proposition and its negation). The CPEM, in its turn, would express that all possible-worlds are complete (no possible-world lacks both a proposition and its negation).

This way of introducing B-negation in FDE₊ suggests the definition of two families of quasi-Boolean negation (QB-negation) expansions of logics including FDE₊. One of them, intuitionistic in character, has the ECQ axiom but not the CPEM one; the other one, dual intuitionistic in nature, has the CPEM axiom, but not the ECQ one. Let us generally refer by H-negation and DH-negation to the former and latter type of negation, respectively ("H" stands for Heyting; "DH", for Dual H-negation).

From a semantical point of view, H-negation is characterized by RM-models composed only by consistent, but not necessarily complete, possible worlds, as it was pointed out above, whereas DH-negation has RM-models with complete, although not necessarily consistent possible-worlds. RM-models where possible-worlds are always consistent and complete determine, of course, B-negation, which, strange as it may seem, can be added to relevant logics included in Anderson and Belnap's logic of the relevant conditional R without breaking down the relevance properties of the positive fragments (cf. Routley et al., 1982, §4 and references therein).

Well then, now it is important to remark that Sylvan and Plumwood's Minimal De Morgan logic B_M is the minimal logic interpretable with RM-semantics (cf. Sylvan & Plumwood, 2003; Routley et al., 1982; Robles & Méndez, 2018). Therefore, we will investigate in the sequel the logic Hb, the minimal extension of B_M with H-negation, and a wealth of extensions of Hb. Concerning these Hb-extensions, we will concentrate on two logics and their subsystems: Gödelian 3-valued logic G3 and H-negation expansion of the 3-valued extension of Lewis' positive modal logic S5₊, S5_{G3} (cf. Definitions A2, A3 and A4 in the appendix). But these two strong logics and many of their subsystems are here treated as a way of an example of how to use the RM-semantics defined in the paper to interpret many other (possibly interesting) extensions of Hb endowed with the type of H-negation we have modelled (in this sense, the paper can be seen as a study on applied non-classical logic). Of course, in none of these logics are the DNE and CPEM axioms provable (cf. the appendix), but the ECQ axiom and the axiom "Double Negation Introduction" (DNI), $A \to \neg \neg A$, are theorems

in all of them (cf. Proposition 2.7). Also, it has to be remarked that the "Principle of Excluded Middle" (PEM), $A \vee \neg A$, intuitionism's *bête noire*, is a theorem of some of the logics included in S5_{G3}. Unreduced Routley-Meyer ternary relational semantics is defined for each one of the logics introduced in the paper (cf. Routley et al., 1982; Brady, 2003 and references therein; cf. also §6).

RM-semantics is a ternary relational semantics which can essentially be divided in two types: (a) RM-semantics with a set of designated points w.r.t. which validity of formulas is decided (RM₁-semantics); (b) RM-semantics without a set of designated points and where validity of formulas is decided w.r.t. the set of all points (RM₀semantics). As for RM₁-semantics, we have reduced RM₁-semantics, where the set of designated points is reduced to a singleton, and unreduced RM₁-semantics. RM₀semantics is not adequate to interpret relevant logics since it necessarily validates paradoxes of relevance. RM₁-semantics, in its turn, can interpret both relevant and non-relevant logics. In the present paper, unreduced RM₁-semantics is provided for all the extensions of Hb we have defined, as remarked above.

The family of logics investigated in the following pages is different from the related families of logics we have previously studied in (Robles & Méndez 2014; Robles & Méndez, 2015; Robles & Méndez, 2018). In Robles & Méndez (2018), both RM₀-semantics and unreduced RM₁-semantics are used, but negation is introduced via a falsity constant instead of using the Routley operator. In (Robles & Méndez, 2014; Robles & Méndez, 2015), RM₀-semantics is the tool we employed. Moreover, the minimal logics considered, $B_{\rm KM}$ in Robles & Méndez (2014) and $H_{\rm M}$ in Robles & Méndez (2015), are not included in many of the subsystems of G3 and in none of those contained in S5_{G3}.

The structure of the paper is as follows. In §2, the minimal logic considered in the paper, Hb, and a wealth of its extensions included in either G3 or $S5_{G3}$ are defined. Hb is the basic extension with an H-negation of Sylvan and Plumwood's minimal De Morgan logic B_M . In §3, unreduced Routley-Meyer semantics is provided for Hb and its extensions defined in §2. In order to define Routley-Meyer semantics for the logics defined in §2, we generally follow the terminology, definitions and strategy of the fundamental work on Routley-Meyer type ternary relational semantics, that is, Chapter 4 of Routley et al. (1982). Weak soundness theorems are proved for all these logics. In §4, we prove some preliminary propositions and lemmas upon which the completeness theorems are established. In §5, (weak) completeness theorems for all the logics defined in §2 are proved. In §6, we point out some remarks on reduced Routley-Meyer semantics and strong completeness. An appendix has been added where some facts stated throughout the paper are proved.

2. The basic logic Hb and its extensions

We begin by defining some basic notions as used in the paper.

Definition 2.1 (Language). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives \rightarrow (conditional), \wedge (conjunction), \vee (disjunction) and \neg (negation). The biconditional (\leftrightarrow) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.

Definition 2.2 (Logics). A logic L is a structure $(\mathcal{L}, \vdash_{\mathrm{L}})$ where \mathcal{L} is a propositional

language and $\vdash_{\mathbf{L}}$ is a (proof-theoretical) consequence relation defined on \mathcal{L} by a set of axioms and a set of rules of inference. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_{\mathbf{L}} A$ means that A is derivable from the set of wffs Γ in L; and $\vdash_{\mathbf{L}} A$ means that A is a theorem of L).

Definition 2.3 (Extensions and expansions of a propositional logic L). Let \mathcal{L} and \mathcal{L}' be two propositional languages. \mathcal{L}' is a strengthening of \mathcal{L} if the set of wffs of \mathcal{L} is a proper subset of the set of wffs of \mathcal{L}' . Next, let L and L' be two logics built upon the propositional languages \mathcal{L} and \mathcal{L}' , respectively. Moreover, suppose that all axioms of L are theorems of L' and all primitive rules of inference of L are provable in L'. Then, L' is an extension of L if \mathcal{L} and \mathcal{L}' are the same propositional language; and L' is an expansion of L if \mathcal{L}' is an strengthening of \mathcal{L} . An extension L' of L is a proper extension if L is not an extension of L'.

The minimal logic considered in the paper is the logic Hb, the basic extension with H-negation of Sylvan and Plumwood's minimal logic B_M .

Definition 2.4 (The logic B_M). Sylvan and Plumwood's minimal logic B_M can be axiomatized with the following axions and rules of inference (cf. Sylvan & Plumwood, 2003):

 $\begin{array}{l} Axioms: (A1) \ A \to A; (A2) \ (A \land B) \to A \ / \ (A \land B) \to B; (A3) \ [(A \to B) \land (A \to C)] \to [A \to (B \land C)]; (A4) \ A \to (A \lor B) \ / \ B \to (A \lor B); (A5) \ [(A \to C) \land (B \to C)] \to [(A \lor B) \to C]; (A6) \ [A \land (B \lor C)] \to [(A \land B) \lor (A \land C)]; (A7) \ (\neg A \land \neg B) \to \neg (A \lor B); (A8) \ \neg (A \land B) \to (\neg A \lor \neg B). \end{array}$

Rules of inference: (Adjunction —Adj) $A \& B \Rightarrow A \land B$; (Modus Ponens —MP) $A \rightarrow B \& A \Rightarrow B$; (Suffixing —Suf) $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$; (Prefixing — Pref) $B \rightarrow C \Rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$; (Contraposition —Con) $A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$. We point out that Routley and Meyer's basic positive logic B_+ can be axiomatized

with A1-A6, Adj, MP, Suf and Pref (cf. Routley et al., 1982).

We note the following remark:

Remark 2.5 (The De Morgan laws). The De Morgan laws (T1) $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$ and (T2) $\neg(A \land B) \leftrightarrow (\neg A \lor \neg B)$ are provable in B_M (by A2-A5, A7, A8 and Con; cf. Sylvan & Plumwood, 2003).

The logic Hb is defined as follows.

Definition 2.6 (The basic logic Hb). The basic logic Hb is axiomatized by adding (A9) $C \to [B \to \neg (A \land \neg A)]$ and (A10) $C \to [(A \land \neg A) \to B]$ to B_M.

We note that the ECQ axiom $(A \land \neg A) \to B$ is not sufficient for axiomatizing Hnegation in weak logics (cf. Proposition 2.9 below and §4), but it could be sufficient if more connectives were added to the formal language of Hb (cf. §7: 'Concluding Remarks'). Also, we remark the following proposition.

Proposition 2.7 (Some theorems of Hb). The following are provable in Hb: (T3) (ECQ axiom) $(A \land \neg A) \rightarrow B$; (T4) Principle of Non-Contradiction (PNC) $\neg (A \land \neg A)$; (T5) Principle of Testability (PTE) $\neg A \lor \neg \neg A$; (T6) (DNI) $A \rightarrow \neg \neg A$.

Proof. T3 and T4 are immediate by A10 and A9, respectively; and T5, by T2 and T4. Then, T6 is proved as follows: we have (1) $A \rightarrow \neg(A \land \neg A)$, by A9, and (2) $(A \land \neg A) \rightarrow \neg \neg A$ by A10. Then, (3) $\neg(A \land \neg A) \rightarrow (\neg A \lor \neg \neg A)$ by T2, and next (4)

 $A \to (\neg A \lor \neg \neg A)$ by 1 and 3. Now we use (5) $A \to [(A \lor \neg \neg A) \land (\neg A \lor \neg \neg A)]$ and (6) $[\neg \neg A \lor (A \land \neg A)] \to (\neg \neg A \lor \neg \neg A)$, which are immediately provable by B_+ , 4 and 2, respectively. Finally, we get (7) $A \to (\neg \neg A \lor \neg \neg A)$ by 5 and 6, and then (8) $A \to \neg \neg A$ by 7 and B_+ .

In what follows we elaborate on the relationship between A9 and A10 and the insufficiency of the ECQ axiom, $(A \land \neg A) \rightarrow B$, for axiomatizing H-negation in certain logics. In Proposition 2.8 it is proved that A9 and A10 are independent in the context of Routley and Meyer's basic logic B. In Proposition 2.9, it is shown that A9 and A10 are not derivable from Anderson and Belnap's *logic of entailment* E plus the ECQ axiom. In Proposition 2.11, it is proved that A9 and A10 are theorems of DW plus the ECQ and Assertion axioms. By a1, a2, ..., a50 we refer to the items in Definition 2.12 below.

Proposition 2.8 (Independence of A9 and A10 given B). A9 and A10 are independent in the context of Routley and Meyer's basic logic B (B is the result of adding the double negation axioms, $A \to \neg \neg A$ and $\neg \neg A \to A$, to B_M —notice that A7 and A8 are then not independent).

Proof. (1) A10 is not derivable from B plus A9. Consider the following set of truth-tables t1 (designated values are starred in this set and in the sets of truth-tables to follow):

\rightarrow	0	1	2	3	\wedge	0	1	2	3	\vee	0	1	2	3		¬
0	2	2	2	3	0	0	0	0	0	 0	0	1	2	3	0	3
1	0	2	0	3	1	0	1	0	1	1	1	1	3	3	1	2
*2	0	0	2	3	*2	0	0	2	2	*2	2	3	2	3	*2	1
*3	0	0	0	3	*3	0	1	2	3	*3	3	3	3	3	*3	0

This table verifies B plus A9 but falsifies $r \to [(p \land \neg p) \to q]$ for any assignment v such that v(p) = 1 = v(q) = v(r) = 1. $(p, q, r \text{ are distinct propositional variables in this proof and the proofs to follow.)$

(2) A9 is not derivable from B plus A10. We use the following set of truth tables t2: tables for \wedge, \vee and \neg are as in the set t1 and the table for \rightarrow is as follows:

\rightarrow	0	1	2	3
0	3	3	3	3
1	0	2	0	3
*2	0	0	2	2
*3	0	0	0	2

This set verifies B plus A10 but falsifies $r \to [q \to \neg(p \land \neg p)]$ for any assignment v such that v(p) = v(q) = v(r) = 3.

Proposition 2.9 (A9 and A10 are not provable in E plus ECQ). A9 and A10 are not provable in E plus the ECQ axiom, $(A \land \neg A) \rightarrow B$. (E is Anderson and Belnap's logic of entailment and can be axiomatized by adding to B - cf. Proposition 2.8— a2, a5, a10, a35 and a36. Notice that the rule Con is then not independent).

Proof. Consider the following set of truth-tables t3: tables for \land, \lor and \neg are as in the set t1, and the table for \rightarrow is as follows:

\rightarrow	0	1	2	3
0	2	2	2	2
1	0	2	0	2
*2	0	0	2	2
*3	0	0	0	2

This set t3 verifies E plus the ECQ axiom, but falsifies $r \to [q \to \neg (p \land \neg p)]$ and $r \to [(p \land \neg p) \to q]$ for any assignment v such that v(p) = v(q) = v(r) = 1. \Box

We note the following remark on Propositions 2.8 and 2.9.

Remark 2.10 (On Propositions 2.8 and 2.9). We note that the result in Proposition 2.8 holds for some considerable extensions of B: sets of truth-tables t1 and t2 verify such theses as a1, a4, a5, a10, a11, a32', a33, a34, a35, a49 and a50. In addition, t1 verifies a18, and t2, a17. On the other hand, t3 can be used to show that the result in Proposition 2.9 can be extended to some considerable extensions of E such as E-Mingle (cf. Anderson & Belnap, 1975) or those built by any selection of the following axioms (verified by t3): the characteristic S3-axiom, $(A \to B) \to [(C \to D) \to (A \to B)]$, the CPEM axiom, $B \to (A \lor \neg A)$, a17, a18 and a32', restricted disjunctive Peirce law, characteristic of the positive fragment of Lewis' S5 (cf. Hacking, 1963).

Proposition 2.11 (A9, A10 are prov. in DW plus ECQ ax. & a14). A9 and A10 are provable in DW plus the ECQ axiom, $(A \land \neg A) \rightarrow B$, and the Assertion axiom a14, $A \rightarrow [(A \rightarrow B) \rightarrow B]$ (DW is axiomatized by adding a36 to B —notice that the rule Con is then derivable.)

Proof. (1) A10 is provable by $(A \land \neg A) \to (C \to B)$ and $C \to [(C \to B) \to B]$; (2) A9 is then provable by A10, the contraposition axiom a36 and the DNI axiom.

The section is ended by introducing the set of extensions of Hb considered in the present paper. These extensions are defined from the axioms and rule displayed in Definition 2.12 below. All axioms and the rule are provable in G3 and/or $S5_{G3}$ (cf. Definitions A2, A3 and A4 in the appendix). Unreduced Routley-Meyer semantics w.r.t. which each one of these extensions is (weakly) sound and complete is provided in the following section.

Definition 2.12 (A set of theses and a rule of G3 and/or $S5_{G3}$). The following theses and rule are provable in G3 and/or $S5_{G3}$: Axioms a1-a31, a33-a48 are provable in Gödelian 3-valued logic G3, while a1-a13, a17, a20-a26, a32, a32', a33-a45, a48-a50 in 3-valued logic $S5_{G3}$.

a1.
$$[(A \to B) \land (B \to C)] \to (A \to C)$$

a2. $(B \to C) \to [(A \to B) \to (A \to C)]$
a3. $(A \to B) \to [(B \to C) \to (A \to C)]$
a4. $[A \land (A \to B)] \to B$
a5. $[A \to (A \to B)] \to (A \to B)$
a6. $A \to [[A \to (A \to B)] \to B]$
a7. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
a8. $(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$
a9. $[A \to (B \to C)] \to [(A \land B) \to C]$

a10.
$$[[(A \to A) \land (B \to B)] \to C] \to C$$

a11. $A \Rightarrow (A \to B) \to B$
a12. $A \to [[A \to (B \to C)] \to (B \to C)]$
a13.
$$[A \to [B \to (C \to D)]] \to [B \to [A \to (C \to D)]]$$

a14. $A \to [(A \to B) \to B]$
a15.
$$[A \to (B \to C)] \to [B \to (A \to C)]$$

a16.
$$(A \land B) \to [[A \land (B \to C)] \to C]$$

a17.
$$(A \to B) \to [[A \land (B \to C)] \to C]$$

a18.
$$[A \land (B \to C)] \to [(A \to B) \to C)]$$

a20. $A \to (A \to A)$
a21. $A \to [B \to (A \lor B)]$
a22.
$$(A \to B) \lor (B \to A)$$

a23.
$$[A \to (B \lor C)] \to [(A \to B) \lor (A \to C)]$$

a24.
$$[(A \land B) \to C] \to [(A \to C) \lor (B \to C)]$$

a25. $B \to (A \to A)$
a26.
$$(A \to B) \to [C \to (A \to B)]$$

a27. $A \to (B \to A)$
a28. $A \to [B \to (C \to A)]$
a29.
$$(A \lor B) \to [(A \to B) \to B]$$

a30. $A \to [B \to (A \land B)]$
a31.
$$[(A \land B) \to C] \to [A \to (B \to C)]$$

a32. $A \lor (A \to B)$
a32.
$$(A \to B) \to [(A \to B) \to C]$$

a33.
$$((A \to B) \lor ((A \to B) \to C])$$

a33.
$$((A \to B) \to (A \to B))$$

a35.
$$((A \to A) \to (A \to B))$$

a36.
$$((A \to B) \to (A \to B))$$

a37.
$$A \to (A \to B)$$

a38.
$$\neg A \to (A \to B)$$

a38.
$$\neg A \to (A \to B)$$

a39.
$$\neg B \to [\neg A \lor (A \to B)]$$

a40.
$$(A \lor B) \lor (A \to B)$$

a41.
$$(\neg A \land B) \to (A \to B))$$

a42.
$$\neg (A \to B) \to (A \to B))$$

a43.
$$[\neg (A \to B) \to (A \to B)]$$

a43.
$$[\neg (A \to B) \to (A \to B)]$$

a44.
$$\neg (A \to B) \to (A \to B))$$

a45.
$$\neg (A \to B) \to (A \to B)]$$

a45.
$$\neg (A \to B) \to (A \to B)]$$

a46.
$$\neg (A \to B) \to (A \to B)]$$

a47.
$$(A \lor B) \to (B \to A))$$

a46.
$$\neg (A \to B) \to (B \to A)]$$

a46.
$$\neg (A \to B) \to (B \to A)]$$

a47.
$$(A \lor B) \to (B \to A)]$$

a48.
$$(A \lor \neg B) \to (\neg A \to \neg B)$$

a49. $A \to [B \lor \neg (A \to B)]$
a50. $A \lor \neg A$

Concerning the extensions of Hb just defined, we note the following remark.

Remark 2.13 (On H-negation extensions of B). Notice that Routley and Meyer's basic logic B (cf. Proposition 2.8) cannot be extended with H-negation on pain of collapse into B-negation: by using the DNE axiom, H-negation immediately collapses into B-negation.

3. RM-semantics for extensions of Hb

In what follows, by an EHb-logic we mean an extension of the basic logic Hb with some subset of a1-a50 in Definition 2.12 provable in either G3 or else in $S5_{G3}$. Notice that B_+ plus a7, a27, a36 and a37 is an axiomatization of propositional intuitionistic logic, H, included in G3. But addition of a50 to H axiomatizes classical propositional logic. We begin by defining EHb-models for extensions of Hb together with the accompanying definitions of truth and validity.

Definition 3.1 (EHb-models). An EHb-model, M, is a structure with at least the following items: (a) A set K and a subset of it, O. (b) A ternary relation R and a unary operation * defined on K subject at least to the following definitions and postulates for all $a, b, c, d \in K$: (d1) $a \leq b =_{df} \exists x \in O Rxab$; (d1') $a = b =_{df} a \leq b \& b \leq a$; (d2) $R^2abcd =_{df} \exists x \in K(Rabx \& Rxcd)$; (P1) $a \leq a$; (P2a) $(a \leq b \& Rbcd) \Rightarrow Racd$; (P2b) $(a \leq b \& b \leq c) \Rightarrow a \leq c$; (P2c) $(d \leq b \& Rabc) \Rightarrow Radc$; (P2d) $(c \leq d \& Rabc) \Rightarrow Rabd$; (P3) $a \leq b \Rightarrow b^* \leq a^*$; (P4) $a \leq a^*$. (c) A valuation relation \vDash from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$: (i) $(a \leq b \& a \models p) \Rightarrow b \models p$; (ii) $a \models A \land B$ iff $a \models A$ and $a \models B$; (iii) $a \models A \lor B$ iff $a \models A \rightarrow B$ iff for all $b, c \in K$, $(Rabc \& b \models A) \Rightarrow c \models B$; (v) $a \models \neg A$ iff $a^* \nvDash A$.

Additional elements of M are a set of semantical postulates $Pj_1, ..., Pj_n$.

Structures of the form $(O, K, R, *, \vDash)$ satisfying d1, d1', d2, P1, P2a, P2b, P2c, P2d, P3, P4 and clauses (i), (ii), (iii), (iv) and (v) are the basic structures and in fact characterize the logic Hb (they are labelled Hb-models). Introduction of additional postulates serves to determine extensions and expansions of Hb interpretable in unreduced Routley-Meyer semantics.

Definition 3.2 (Truth). Let a class of EHb-models \mathcal{M} be defined and $M \in \mathcal{M}$. A wff A is true in M (in symbols, $\vDash_M A$) iff $a \vDash A$ for all $a \in O$.

Definition 3.3 (Validity). Let a class of EHb-models \mathcal{M} be defined. A wff A is valid in \mathcal{M} (in symbols, $\vDash_{\mathcal{M}} A$) iff A is true in every $M \in \mathcal{M}$.

We note the following proposition.

Proposition 3.4 (A couple of semantical postulates). Let a class of EHb-models \mathcal{M} be defined. Then, the following semantical postulates P4a and P4b are provable in any $M \in \mathcal{M}$: (P4a) $a^* \leq a^{**}$; (P4b) $a \leq a^{**}$.

Proof. P4a is immediate by P4; P4b follows immediately by P4, P4a and P2b. \Box

Now, we need to prove that all theorems of Hb are EHb-valid, i.e., valid in any class of EHb-models \mathcal{M} . But, given weak soundness of B_M w.r.t. EB_M -models (cf. Sylvan & Plumwood, 2003) and, so, w.r.t. EHb-models, it is clear that we only have to show the EHb-validity of A9 and A10 in order to prove that all theorems of Hb are EHb-valid. The following two useful lemmas are employed without mentioning them in the soundness proofs to follow.

Lemma 3.5 (Hereditary Lemma). For any EHb-model and $a, b \in K$ and wff A, $(a \leq b \& a \models A) \Rightarrow b \models A$.

Proof. Induction on the length of A. The conditional case is proved with P2a and the negation case is proved with P3. \Box

Lemma 3.6 (Entailment Lemma). Let a class of EHb-models \mathcal{M} be defined. For any wffs $A, B, \vDash_{\mathcal{M}} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$ for all $a \in K$) in all $M \in \mathcal{M}$.

Proof. From left to right, by P1; from right to left, by Lemma 3.5.

Proposition 3.7 (A9 and A10 are EHb-valid). Let \mathcal{M} be a class of EHb-models. Then, A9 and A10 are valid in \mathcal{M} .

Proof. Let $M \in \mathcal{M}$. We prove that A9 and A10 are true in M.

(a) $A9, C \to [B \to \neg(A \land \neg A)]$, is true in M: For reductio, suppose that there are wffs A, B, C and $a \in K$ in M such that (1) $a \models C$ but (2) $a \nvDash B \to \neg(A \land \neg A)$. Then, we have $b, c \in K$ in M such that (3) Rabc, (4) $b \models B$, (5) $c \nvDash \neg(A \land \neg A)$, i.e., (6) $c^* \models A$ and (7) $c^* \models \neg A$. By 7 we get (8) $c^{**} \nvDash A$. But by P4a and 6, we have (9) $c^{**} \models A$, contradicting 8.

(b) $A10, C \to [(A \land \neg A) \to B]$, is true in M: For reductio, suppose that there are wffs A, B, C and $a \in K$ in M such that (1) $a \models C$ but (2) $a \nvDash (A \land \neg A) \to B$. Then, there are $b, c \in K$ in M such that (3) Rabc, (4) $b \models A \land \neg A$ and $c \nvDash B$. Given 4, we have (5) $b \models A$ and (6) $b^* \nvDash A$, which is impossible by P4.

Once Proposition 3.7 is proved, we immediately have the following corollary.

Corollary 3.8 (All theorems of Hb are EHb-valid). For any wff A, if $\vdash_{Hb} A$, then A is EHb-valid (i.e., valid in any class of EHb-models).

Proof. Immediate, given soundness of B_M w.r.t. EB_M -models (cf. Sylvan & Plumwood, 2003; cf. Definition 3.1 and Proposition 3.7).

Corollary 3.9 (Soundness of Hb). For any wff A, if $\vdash_{Hb} A$, then $\models_{Hb} A$.

Proof. Immediate by Corollary 3.8, since an Hb-model is an EHb-model.

In what follows, we proceed to prove the soundness of the EHb-logics (recall that by an EHb-logic we mean an extension of Hb with some subset of the theses in Definition 2.12, provable in either G3 or $S5_{G3}$). The basic notion is "corresponding postulate" (cf. Routley et al., 1982, Chapter 4). We give a corresponding semantical postulate to each one of the items a1 through a50. Then, given Corollary 3.8, a class of EHb-models \mathcal{M} , and any $M \in \mathcal{M}$, we only need to prove that a_k ($1 \le k \le 50$) is true in M (or preserves truth in M, as the case may be) provided its corresponding semantic postulate pa_k holds in M.

Let now L be any EHb-logic. The section is ended by the definition of L-models and the proof of (weak) soundness of L w.r.t. L-models. L-models are simply defined by adding to Hb-models the corresponding postulates to the items in Definition 2.12 added to build L from Hb. Then, soundness of L is immediate from Corollary 3.8 (all theorems of Hb are EHb-valid) and Lemma 3.11 (EHb-validity of pa1-pa50).

Definition 3.10 (Postulates corresponding to a1-a50). Below, we provide postulates corresponding to each one of a1-a50.

pa1. $Rabc \Rightarrow \exists x (Rabx \& Raxc)$ pa2. $R^2abcd \Rightarrow \exists x (Rbcx \& Raxd)$ pa3. $R^2abcd \Rightarrow \exists x(Racx \& Rbxd)$ pa4. Raaa pa5. $Rabc \Rightarrow R^2 abbc$ pa6. $Rabc \Rightarrow R^2 baac$ pa7. $R^2abcd \Rightarrow \exists x, y(Racx \& Rbcy \& Rxyd)$ pa8. $R^2abcd \Rightarrow \exists x, y(Racx \& Rbcy \& Ryxd)$ pa9. $Rabc \Rightarrow R^2 abbc$ pa10. $\exists x \in Z \ Raxa \ [Za \text{ iff for all } b, c \in K, Rabc \Rightarrow \exists x \in O \ Rxbc]$ pa11. $\exists x \in O \ Raxa$ pa12. $R^2 abcd \Rightarrow R^2 bacd$ pa13. $R^3abcde \Rightarrow R^3acbde$ pa14. $Rabc \Rightarrow Rbac$ pa15. $R^2abcd \Rightarrow R^2acbd$ pa16. $Rabc \Rightarrow R^2 baac$ pa17. $Rabc \Rightarrow \exists x (Rabx \& Rbxc)$ pa18. $Rabc \Rightarrow \exists x (Rbax \& Raxc)$ pa19. $R^2abcd \Rightarrow R^2bcad$ pa20. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$ pa21. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$ pa22. (*Rabc & Rade & a* $\in O$) \Rightarrow (*b* $\leq e$ or *d* $\leq c$) pa23. (Rabc & Rade) $\Rightarrow \exists x [(Rabx \text{ or } Radx) \& x \leq c \& x \leq e]$ pa24. (Rabe & Rade) $\Rightarrow \exists x [(Raxc \text{ or } Raxe) \& b \leq x \& d \leq x]$ pa25. $Rabc \Rightarrow b \leq c$ pa26. $R^2abcd \Rightarrow Racd$ pa27. $Rabc \Rightarrow a \leq c$ pa28. $R^2 abcd \Rightarrow a \leq d$ pa29. $Rabc \Rightarrow (Rbac \& a \leq c)$ pa30. $Rabc \Rightarrow (a \leq c \& b \leq c)$ pa31. $R^2abcd \Rightarrow \exists x (Raxd \& b < x \& c < x)$ pa32. (Rabc & $a \in O$) $\Rightarrow b \leq a$ pa32'. (Rabc & Rade & $a \in O$) \Rightarrow Rdbc

pa33. Raa^*a^* pa34. Ra^*aa^* pa35. Raa^*a^{**} pa36. $Rabc \Rightarrow Rac^*b^*$ pa37. $Rabc \Rightarrow a < b^*$ pa38. $Rabc \Rightarrow b \leq a^*$ pa39. $Ra^*a^*a^*$ pa40. (*Rabc* & $a \in O$) \Rightarrow ($b \leq a \text{ or } a \leq c^*$) pa41. $Rabc \Rightarrow (b \leq a^* \text{ or } a \leq c)$ pa42. $Ra^*bc \Rightarrow (b \le a \text{ or } a^* \le c)$ pa43. $Ra^*bc \Rightarrow (b \le a^* \text{ or } a \le c)$ pa44. $Rabc \Rightarrow c^* < a^*$ pa45. (Rabc & Ra^{*}de) \Rightarrow ($d \leq c$ or $b \leq e$) pa46. $Ra^*bc \Rightarrow a^* \leq c$ pa47. $Rabc \Rightarrow (a \leq b^* \text{ or } a \leq c)$ pa48. $Rabc \Rightarrow (a < b^* \text{ or } c^* < a^*)$ pa49. Ra^*aa pa50. $a \in O \Rightarrow a^* < a$

Lemma 3.11 (EHb-validity of a1-a50). Let \mathcal{M} be a class of EHb-models and $M \in \mathcal{M}$. Then, for any k ($1 \le k \le 10$; or $12 \le k \le 50$) ak is true in M if pak holds in M; and all preserves truth in M if pall holds in M.

Proof. The proof of the validity of a1-a32, a32' (that a11 preserves truth in M, in the case of a11) can be found in Robles & Méndez (2018). The proof of a33-50 is similar to that given in Routley et al. (1982, Chapter 4), for extensions of Routley and Meyer's basic logic B. Let us prove some cases:

(a) a34, $(A \land \neg B) \to \neg (A \to B)$, is true in M: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \models A \land \neg B$ but (2) $a \nvDash \neg (A \to B)$. By 1, we have (3) $a \models A$ and (4) $a^* \nvDash B$; and by 2, (5) $a^* \models A \to B$. Then, we apply pa34 (6) Ra^*aa^* , and we conclude (7) $a^* \models B$ by 3, 5 and 6, contradicting 4.

(b) a38, $\neg A \rightarrow (A \rightarrow B)$, is true in M: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \models \neg A$ but (2) $a \nvDash A \rightarrow B$. By 1, we get (3) $a^* \nvDash A$, and by 2, there are $b, c \in K$ in M such that (4) Rabc, (5) $b \models A$ and (6) $c \nvDash B$. By pa38 and 4, we get (7) $b \leq a^*$, whence by 5, we obtain (8) $a^* \vDash A$, contradicting 3.

(c) a40, $(A \lor \neg B) \lor (A \to B)$, is true in M: for reductio suppose that there are wffs A, B and $a \in O$ such that (1) $a \nvDash A$, (2) $a \nvDash \neg B$ (i.e, $a^* \vDash B$) and (3) $a \nvDash A \to B$. By 3, there are $b, c \in K$ such that (4) Rabc, (5) $b \vDash A$ and (6) $c \nvDash B$. By pa40 and 4 (given that $a \in O$), we have (7) $b \le a$ or (8) $a^* \le c$. But 5 and 7 contradict 1, whereas 2 and 8 contradict 6.

(d) a45, $\neg(A \to B) \to (B \to A)$, is true in M: For reductio, suppose that there are wffs A, B and $a \in K$ in M such that (1) $a \models \neg(A \to B)$, i.e., (2) $a^* \nvDash A \to B$, but (3) $a \nvDash B \to A$. By 2, there are $b, c \in K$ in M such that (4) Ra^*bc , (5) $b \vDash A$ and (6) $c \nvDash B$. By 3, there are $d, e \in K$ in M such that (7) Rade, (8) $d \vDash B$ and (9) $e \nvDash A$. But by pa45, we have either (10) $d \le c$ or (11) $b \le e$. Then, 8 and 10 contradict 6; and 5 and 11 contradict 9.

Definition 3.12 (L-models). Let L be an EHb-logic. An L-model is defined when adding to Hb-models the semantical postulates corresponding to the items in Definition 2.12 added to Hb for axiomatizing L. For example, consider the extension of Hb axiomatized by a7 and a26 Then, an EHb-model for this system is a structure $(O, K, R, *, \vDash)$ where O, K, R, *, and \vDash are defined exactly as in Definition 3.1, save for the addition of the postulates pa7 and pa26. (The notion of L-validity is defined according to the general Definition 3.3. Notice that the system just defined is the expansion of the positive fragment of Lewis' logic S4 (cf. Hacking, 1963) with the basic quasi-Boolean H-negation defined above).

Theorem 3.13 (Soundness of EHb-logics). Let L be an EHb-logic. For any wff A, if $\vdash_L A$, then $\models_L A$.

Proof. By Corollary 3.8 and Lemma 3.11, given Definition 3.12.

The section is ended with the following remark (cf. Remark 2.13).

Remark 3.14 (Bb-models). We note that Bb-models, RM-models for B-negation, are defined by adding the postulate $a^* \leq a$ to Hb-models. Bb-models characterize Hb plus the axioms $C \to [B \to (A \lor \neg A)]$ (A9') and $C \to [\neg(A \lor \neg A) \to B]$ (A10').

4. Completeness. Preliminary notions and lemmas

We use a canonical model construction in order to prove the completeness of the EHblogics. The canonical model is defined and then we prove that if A is not a theorem of the logic in question, then A is false in at least a designated point in the canonical model. Firstly, we define some preliminary notions needed to define the canonical model.

Definition 4.1 (EHb-theories). Let L be an EHb-logic. An L-theory is a set of wffs closed under Adjunction (Adj) and L-entailment (L-ent). That is, a is an L-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of L and $A \in a$, then $B \in a$.

By the term EHb-theory, we will generally refer to any theory defined upon an EHblogic as just indicated. The classes of EHb-theories of interest in the present paper are remarked in the following definition.

Definition 4.2 (Classes of EHb-theories). Let L be an EHb-logic and a an L-theory. We set: (1) a is prime iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$. (2) a is empty iff it contains no wffs. (3) a is regular iff a contains all theorems of L. (4) a is trivial iff every wff belongs to it. (5) a is a-consistent (consistent in an absolute sense) iff a is not trivial.

Next, we prove two useful propositions.

Proposition 4.3 (On non-empty EHb-theories). Let L be an EHb-logic and a a nonempty L-theory. Then, $\neg(A \land \neg A) \in a$ for any wff A.

Proof. Immediate by A9.

Proposition 4.4 (a is a-inconsistent iff $A \wedge \neg A \in a$). Let L be an EHb-logic and a

an L-theory. Then, a is a-inconsistent iff $A \wedge \neg A \in a$ for some wff A.

Proof. Immediate by T3 (ECQ).

Notice then that, within the context of Hb and its extensions, a-consistency and (negation) consistency in the customary sense of the term coincide.

Next, canonical models are defined.

Definition 4.5 (Canonical EHb-models). Let L be an EHb-logic and K^T be the set of all L-theories. Then, the ternary relation R^T is defined in K^T as follows: for any $a, b, c \in K^T$, $R^T abc$ iff for any wffs $A, B, (A \to B \in a \& A \in b) \Rightarrow B \in c$. Next, let K^C be the set of all prime, non-empty and a-consistent L-theories, O^C be the subset of K^C formed by all regular L-theories and R^C be the restriction of R^T to K^C . On the other hand, let $*^C$ be defined on K^C as follows: for all $a \in K^C$, $a^{*^C} = \{A \mid \neg A \notin a\}$. Finally, the relation \models^C is defined as follows: for each formula A and $a \in K^C$, $a \models^C A$ iff $A \in a$. Then, the structure $(O^C, K^C, R^C, *^C, \models^C)$ is the canonical L-model.

As pointed out above, completeness leans upon proving that the canonical L-model is indeed an L-model. Now, the proof of this fact is similar to the corresponding proofs in standard relevant logics (cf. Routley et al., 1982, Chapter 4), except for an important difference: everytime an L-theory a is built, a has to be proven non-empty and a-consistent. But this is easily shown by using Propositions 4.3 and 4.4.

In the sequel, a series of preliminary lemmas follows. We suppose we are given an EHb-logic L: some of the lemmas below are not provable for weaker logics.

Lemma 4.6 (On the relation R^T). Let a, b be non-empty L-theories and c be an a-consistent and prime L-theory. Then,

(a) The set $x = \{B \mid \exists A(A \rightarrow B \in a \& A \in b)\}$ is a non-empty L-theory such that $R^T abx$.

(b) If R^T abc, then there is a prime and a-consistent (and non-empty) prime L-theory x such that $a \subseteq x$ and $R^T x bc$.

(c) If R^T abc, then there is a prime and a-consistent (and non-empty) prime L-theory x such that $b \subseteq x$ and R^T axc.

Proof. (a) It is easy to prove that x is an L-theory such that $R^T abx$. Then, the nonemptiness of x follows by A9 (notice that the simpler thesis $B \to \neg (A \land \neg A)$ is not sufficient).

(b) A prime L-theory a such that $a \subseteq x$ and $R^T x bc$ is built up similarly as in standard relevant logics. Next, x is immediately shown a-consistent. If x is a-inconsistent, then $A \to B \in x$, for $A \in b$ and arbitrary wff B, whence c would be a-inconsistent, contradicting one of the hypotheses.

(c) As in the precedent case, a prime L-theory x such that $b \subseteq x$ and $R^T axc$ is built up similarly as in relevant logics. Then, the a-consistency of x is proven as follows. Suppose x is a-inconsistent. Let $A \in a$ and B be an arbitrary wff. Then, $C \land \neg C \in x$ for some wff C, by Proposition 4.4. By A10, $A \to [(C \land \neg C) \to B]$ is an L-theorem. So, $(C \land \neg C) \to B \in a$, whence $B \in c$, by $R^T axc$, contradicting the a-consistency of c(notice that the ECQ axiom is not sufficient).

Lemma 4.7 ($*^{C}$ is an operation on K^{C}). Let a be a prime, non-empty and aconsistent L-theory. Then, $a^{*^{C}}$ is a prime, non-empty and a-consistent L-theory as well. **Proof.** As there is no danger of confusion between a^* in K and the canonical L-theory a^{*^C} in K^C , we omit the supersript C above * in this proof and the proofs to follow. Now, a^* is proven a prime L-theory similarly as in standard relevant logics. Then a^* is shown non-empty and a-consistent as follows. (1) a^* is non-empty. Let $A \in a$. If a^* is empty, then $A \notin a^*$, whence $A \wedge \neg A \in a$, contradicting the a-consistency of a. (2) a^* is a-consistent. If a^* is a-inconsistent, then $A \wedge \neg A \in a^*$ for some wff A, whence $\neg (A \wedge \neg A) \notin a$, contradicting the non-emptiness of a (cf. Proposition 4.3).

Lemma 4.8 $(a \leq^C b \text{ iff } a \subseteq b)$. For any $a, b \in K^C$, $a \leq^C b \text{ iff } a \subseteq b$. (By d1, $a \leq^C b$ is equivalent to $R^C xab$, for some $x \in O^C$.)

Proof. It is proved similarly as in standard relevant logics, except that in addition to being regular, x has to be proven a-consistent, which is immediate by Lemma 4.6(b).

Finally, we prove the primeness lemma and then the adequacy of the canonical valuation relation.

Lemma 4.9 (Extension to prime L-theories). Let a be an L-theory and A a wff such that $A \notin a$. Then, there is a prime L-theory x such that $a \subseteq x$ and $A \notin x$.

Proof. By direct application of Kuratowski-Zorn's Lemma as in Routley et al. (1982, Chapter 4, pp. 310-311). \Box

Lemma 4.10 (\models^C and clauses (i)-(v)). For any $a, b, c \in K^C$ and wffs A, B: (i) ($a \leq^C b \& a \models^C p$) $\Rightarrow b \models^C p$; (ii) $a \models^C A \land B$ iff $a \models^C A$ and $a \models^C B$; (iii) $a \models^C A \lor B$ iff $a \models^C A$ or $a \models^C B$; (iv) $a \models^C A \to B$ iff for all $b, c \in K^C$, ($R^C abc \& b \models^C A$) $\Rightarrow c \models^C B$; (v) $a \models^C \neg A$ iff $a^{*^C} \nvDash^C A$.

Proof. Similar to those in relevant logics, save that Lemma 4.6 (cases a and c) is used to prove non-emptiness and a-consistency when required. \Box

Notice that Lemma 4.6 (cases a and c) and Lemma 4.10 require A9 and A10 in order to be proven.

5. Completeness of the EHb-logics

By using the primeness lemma (Lemma 4.9), it is easy to prove that if A is not an L-theorem, then there is some prime, regular and a-consistent L-theory not containing A. Then, for proving the completeness of L, it only remains to show that the canonical L-model is in fact a model. Now, given Lemma 4.7 (*^C is an operation on K^{C}) and Lemma 4.10 (Adequacy of the canonical evaluation clauses), it only remains to prove (1) the set O^{C} is non-empty; and (2) the postulates hold canonically.

Corollary 5.1 (O^C is not empty). Let $(O^C, K^C, R^C, *^C, \models^C)$ be the canonical L-model. Then, the set O^C is not empty.

Proof. Clearly, L is a-consistent since all axioms and rules of L are theorems and rules of classical propositional logic when read with the classical connectives. Then, Corollary 5.1 is immediate by Lemma 4.9. \Box

Lemma 5.2 (The postulates are canonically valid). Let L be an EHb-logic. Then, (1) P1, P2a, P2b, P2c, P2d, P3 and P4 hold in all canonical EHb-models. (2) pak holds in the canonical L-model if ak is provable in L $(1 \le k \le 50)$.

Proof. The proof is similar to that provided in Routley et al. (1982, Chapter 4), for extensions of Routley and Meyer's basic logic B. Actually, a proof for P1, P2a, P2b, P2c, P2d and P3 can be found in the aforementioned chapter and P4 is proved below. Then, pak holds in the canonical L-model if ak is provable in L ($1 \le k \le 50$). Now, concerning pa1-pa32 and pa32', the proof can be found in Robles & Méndez (2018). And concerning pa33-pa50, we prove the canonical validity of the postulates used in Lemma 3.11, as a way of an example.

(a) P4, $a \leq a^*$, is provable in the canonical L-model: Suppose $a \in K^C$ and (1) $A \in a$. We have to prove $A \in a^*$. Suppose (2) $A \notin a^*$. Then, (3) $\neg A \in a$ whence by 1, we get (4) $A \land \neg A$, contradicting the a-consistency of a (cf. Proposition 4.4).

(b) pa34, Ra^*aa^* , is provable in the canonical L-model: Let $a \in K^C$ and suppose (1) $A \to B \in a^*$ and (2) $A \in a$. We have to prove $B \in a^*$. By 1, we get (3) $\neg (A \to B) \notin a$, whence, by a34, $(A \land \neg B) \to \neg (A \to B)$, we have (4) $A \land \neg B \notin a$, i.e., (5) $A \notin a$ or $\neg B \notin a$. Then, (6) $\neg B \notin a$ follows by 2 and, finally, (7) $B \in a^*$ by 6, as was to be proved.

(c) pa38, $Rabc \Rightarrow b \leq a^*$ is provable in the canonical L-model: Let $a, b, c \in K^C$ and suppose (1) $R^C abc$ and (2) $A \in b$. We have to prove $A \in a^*$. Let B an arbitrary wff and for reductio suppose (3) $A \notin a^*$, i.e., $\neg A \in a$. By a38 we have (4) $\neg A \rightarrow (A \rightarrow B)$, whence by 3, we get (5) $A \rightarrow B \in a$ and, finally, by 1, 2 and 5, (6) $B \in c$, contradicting the a-consistency of c.

(d) pa40, $(Rabc \& a \in O) \Rightarrow (b \leq a \text{ or } a^* \leq c)$, is provable in the canonical *L*-model: Let $a \in O^C$, $b, c \in K^C$ and suppose (1) $R^C abc$ and, for reductio, (2) $A \in b$, (3) $A \notin a$, (4) $B \in a^*$, i.e., $\neg B \notin a$ and (5) $B \notin c$. By a40, we have (6) $(A \vee \neg B) \vee (A \rightarrow B) \in a$, given that $a \in O^C$. Then, by 3, 4 and 6, we get (7) $A \rightarrow B \in a$, whence by 1 and 2, (8) $B \in c$ is obtained, contradicting 5.

(e) pa45, $(Ra^*bc \& Rade) \Rightarrow (d \leq c \text{ or } b \leq e)$, is provable in the canonical *L*-model: Let $a, b, c \in K^C$ and suppose (1) R^Ca^*bc , (2) Rade and, for reductio, (3) $A \in d$, (4) $A \notin c$, (5) $B \in b$ and (6) $B \notin e$. By 1, 4 and 5, we have (7) $B \to A \notin a^*$, i.e., (8) $\neg(B \to A) \in a$. By a45, (9) $\neg(B \to A) \to (A \to B)$ is an L-theorem. Then, we get (10) $A \to B \in a$, by 8 and 9. Finally, by 2, 3 and 10, (11) $B \in e$ is derivable, contradicting 6.

Proposition 5.3 (The canonical model is a model). Let L be an EHb-logic. The canonical L-model is indeed an L-model.

Proof. Given Definition 4.5 and Corollary 5.1, the proof follows by Lemma 4.7 ($*^{C}$ is a operation on K^{C}), Lemma 4.10 (Adequacy of the canonical clauses) and Lemma 5.2 (The postulates hold canonically).

Theorem 5.4 (Completeness of the EHb-logics). Let L be an EHb-logic. For any wff A, if $\vDash_L A$, then $\vdash_L A$.

Proof. We prove the contrapositive of the claim. Suppose A is a formula such that $\nvDash_{\mathrm{L}} A$ and let L be the set of all its theorems. Then, $A \notin \mathrm{L}$, and by Lemma 4.9, there is a prime and regular (and a-consistent) L-theory x such that $\mathrm{L} \subseteq x$ and $A \notin x$. Then, the canonical model is defined and x is a member of O^C in the canonical L-model such that $x \nvDash^C A$. Given that the canonical L-model is an L-model, we have $\nvDash_{\mathrm{L}} A$ by

6. A note on reduced Routley-Meyer semantics and strong completeness

As pointed out in the introduction of this paper, there are two types of RM-semantics, RM_0 -semantics and RM_1 -semantics. The latter can be reduced or unreduced RM_1 -semantics. In Brady (2003) and Routley et al. (1982), it is explained at length why reduced RM_1 -semantics is preferable when it is possible to define it.

Well then, a necessary condition for defining a reduced RM_1 -semantics for a given logic L is that it is possible to build up prime regular theories closed under the primitive rules of inference of L. But this necessary condition is not generally met by weak logics. Let us elaborate on the question. Consider the logics C, C' defined below (cf. Routley et al., 1982, p. 289 —the logic C' is defined by us).

Definition 6.1 (The logic C). The logic C is axiomatized by adding to B (cf. Proposition 2.8) the following axioms: a3 (Axiom Suffixing), a4 (Axiom Modus Ponens) and a36 (Axiom Contraposition) (cf. Definition 2.12; we note that a2 (Axiom Prefixing) is derivable in C by a3, a36 and the Double Negation Axioms).

Definition 6.2 (The logic C'). The logic C', a sublogic of C, is axiomatized by adding to B_M the axioms a2, a3, a4 and a36.

In Routley et al. (1982, Chapter 4), it is essentially shown that C and any of its extensions with Adj and MP as the sole primitive rules of inference can be given a reduced RM_1 -semantics. Although we cannot prove it here, it could be shown that this result still holds for the weaker logic C' and its extensions with only Adj and MP as primitive rules of inference. Consequently, all extensions of Hb included in G3 and/or $S5_{G3}$ with a2, a3, a4, a36 and Adj and MP as the only primitive rules of inference could be endowed with a reduced RM_1 -semantics instead of the unreduced one provided for them in the present paper. Moreover, even logics weaker than C' can be given a reduced RM₁-semantics on the condition that each one of their primitive rules of inference (except Adj) be accompanied by its disjunctive version or the corresponding "thesis form" (cf. Routley et al., 1982, pp. 356, ff.; for example, the disjunctive version of the rule Suf is $D \lor (A \to B) \Rightarrow D \lor [(B \to C) \to (A \to C)]$, while the corresponding thesis form is, of course, a3). Consequently, any extension L of Hb included in G3 and/or S_{5G3} with any (or all) the rules MP, Suf, Pref, Con and all as primitive rules of inference can be given a reduced RM₁-semantics with the proviso that the disjunctive version or the thesis form of the rules present be added.

Related in a way, that we cannot pause here to explain with some detail, to the possibility of defining a reduced RM₁-semantics for a given logic L is that of proving a standard strong completeness theorem for L. In general, if L is a logic with other primitive rules of inference than Adj and MP and lacking the disjunctive versions or the thesis forms of the primitive rules in question, then strong completeness (of sorts) can only be proven w.r.t. a deductive consequence relation $\vdash_{\rm L}$ defined as follows: for any set of wffs Γ and wff A, $\Gamma \vdash_{\rm L} A$ iff there is a finite sequence of wffs $B_1, ..., B_m$ such that for each B_i $(1 \le i \le m)$, we have (1) $B_i \in \Gamma$; (2) B_i is a theorem of L; (3) B_i is the result of applying the rule Adj; (4) B_i is the result of applying the rule L-entailment (L-ent) to two precedent wffs in the sequence (L-ent is the following rule: $\vdash_{\rm L} A \to B$ & $A \Rightarrow B$). Nevertheless, a standard strong completeness theorem is provable for L provided the disjunctive versions or the thesis forms of the primitive rules of inference of L are added to its axiomatization.

7. Concluding remarks

It was pointed out in Proposition 2.9 and in §4 that axiomatization of H-negation (as this notion has been understood in the present paper) requires A9 and A10 in general, since the ECQ axiom, $(A \land \neg A) \rightarrow B$, is insufficient in the case of certain logics. Nevertheless, the ECQ axiom could be sufficient in logics built on expansions of the language upon which Hb and its extensions have been defined here. In particular, if fusion (\circ) and the 'left implication' (\leftarrow) of the Lambeck calculus (cf. e.g., Restall, 2000a) are added as new connectives, then A9 and A10 can be derived from B_M plus the ECQ axiom (we owe this remark to an anonymous referee of the JANCL). Maybe this fact would also obtain in the case of addition of other connectives, such as those investigated in Bimbó & Dunn (2008). Thus, future work on the topic could consist in the restructuration of the systems investigated above now based on a new definition of Hb built upon the expanded language with \circ and \leftarrow , and/or with other connectives as well.

In the present paper, we have defined a family of logics including Sylvan and Plumwood's logic B_M extended with an intuitionistic negation of sorts, essentially axiomatized by A9 and A10. Of course, there is an immense literature on the two topics central to the investigation here reported, intuitionistic logics and the notion of negation (cf., e.g., Moschovakis, 2018 and Horn & Wansing, 2020, and references in these two items), but we think that the perspective we have adopted is still new. Let us briefly comment on work related to the one recorded in the preceding pages. Generally speaking, the literature can be divided in two types of investigations: (a) negation expansions of systems built upon a minimal consequence relation and without using the conditional connective. For instance, one of the aforesaid minimal systems can be found in Dunn (2000). It has the rule $A \wedge \neg A \vdash B$, or alternatively, $B \vdash A \vee \neg A$. Or to take another example, in Shramko (2005), the rules assumed are $A \wedge \neg A \vdash B$ and $A \vdash \neg \neg A$, or alternatively, $B \vdash A \lor \neg A$ and $\neg \neg A \vdash A$ (cf. also Restall, 2000b; Restall, 1999; and Dunn, 1993); (b) negation expansions built upon positive systems with a strong conditional. For instance, the positive fragments of relevant logic, intuitionistic logic and both Gödelian 3-valued logic G3 and Lewis' S5 are the starting point in Dunn (2000), Wansing (2016) and Yang (2012), respectively. In this sense, we think that the point of view we have adopted in the present paper is not only new within the semantics used (RM-semantics), but also in what concerns the family of logics studied with the said semantics.

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Appendix A. Appendix

Proposition A.1 (Ant, DNE are deriv. in FDE₊ plus ECQ & CPEM). The axiom $DNE, \neg \neg A \rightarrow A$, and the rule Ant, $(A \wedge B) \rightarrow \neg C \Rightarrow (A \wedge C) \rightarrow \neg B$ are derivable in FDE_+ plus the axioms ECQ, $(A \wedge \neg A) \rightarrow B$, and CPEM, $B \rightarrow (A \vee \neg A)$.

Proof. (Sketch)

(a) $(A \land B) \to \neg C \Rightarrow (A \land C) \to \neg B$:

Suppose (1) $(A \wedge B) \to \neg C$ (Hyp) and (2) $(C \wedge \neg C) \to \neg B$ (ECQ). By 1, 2 and FDE₊, we have (3) $[(A \wedge C) \wedge B] \to \neg B$. On the other hand, we obviously have (4) $[(A \wedge C) \wedge \neg B] \to \neg B$. By 3, 4 and FDE₊, we get (5) $[(A \wedge C) \wedge (B \vee \neg B)] \to \neg B$. Now, we use (6) $C \to (B \vee \neg B)$ (CPEM), whence we obtain (7) $[(A \wedge C) \wedge (A \wedge C)] \to [(A \wedge C) \wedge (B \vee \neg B)]$. Finally, by 5, 7 and FDE₊, we get (8) $(A \wedge C) \to \neg B$, as was to be proved.

(b) $\neg \neg A \rightarrow A$:

We have $(1) \neg \neg A \rightarrow (A \lor \neg A)$ (CPEM). By 1 and FDE₊, we have $(2) (\neg \neg A \land \neg \neg A) \rightarrow [(\neg \neg A \land A) \lor (\neg \neg A \land \neg A]$. We use now $(3) (\neg \neg A \land \neg A) \rightarrow A$ (ECQ). By 3 and FDE₊, we get $(4) [(\neg \neg A \land A) \lor (\neg \neg A \land \neg A)] \rightarrow A$. Finally, by 2, 4 and FDE₊, we have $(5) \neg \neg A \rightarrow A$, as was to be proved.

Consider the following definitions.

Definition A.2 (The matrix MG3). The matrix MG3 is the structure $(\mathcal{V}, D, \mathcal{F})$ where $(1) \mathcal{V} = \{0, 1, 2\}$ and 0 < 1 < 2; $(2) D = \{2\}$ and $\mathcal{F} = \{f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}\}$ where $f_{\rightarrow}, f_{\vee}, f_{\neg}$ are defined according to the following truth-tables:

\rightarrow	0	1	2	\wedge	0	1	2		V	0	1	2		¬
0	2	2	2	0	0	0	0	-	0	0	1	2	0	2
1	0	2	2	1	0	1	1		1	1	1	2	1	0
2	0	1	2	2	0	1	2		2	2	2	2	2	0

Definition A.3 (The matrix $MS5_{G3}$). The matrix $MS5_{G3}$ is the structure $(\mathcal{V}, D, \mathcal{F})$ where \mathcal{V}, D and \mathcal{F} are defined similarly as in MG3, except that now $D = \{1, 2\}$ and f_{\rightarrow} is defined according to the following truth-table:

\rightarrow	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

Notice that CPEM and DNI are falsified by both MG3 and $MS5_{G3}$.

Definition A.4 (Axiomatization of G3 and S5_{G3}). Positive intuitionistic logic, H_+ , can be axiomatized by adding a7 and a27 to B_+ (cf. Definitions 2.4 and 2.12); on the other hand, positive modal logic S5 can be axiomatized by adding a7, a26 and a32' to B_+ (cf. Hacking, 1963). Then, G3 is axiomatized as follows: H_+ plus DNI, a36, a38 and a40 (cf. Robles, 2014, and references therein). Concerning the logic S5_{G3}, that is, the logic determined by MS5_{G3}, we have not axiomatized it and we ignore if it has been axiomatized somewhere in the literature. However, we remark that two logics related to it, the logics determined by the matrices resulting of changing the truth-table for negation in MS5_{G3} for the ones below have been axiomatized in Robles & Méndez (2019) and Yang (2012), respectively.

	¬		-
0	2	0	2
1	1	1	2
2	0	2	0